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MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 4, May 27, 2014[1]

1. Two pizza places differ in how they prefer to box their pizzas. One produces circular pizzas and delivers them in a square box, while the other produces square pizzas and delivers them in a circular box. Who is wasting a higher proportion of their box space?

2. Bec stands at the base of a flight of ten stairs. Her little legs can only take either one or two steps at a time. In how many different ways can she ascend the stairs?

3. A common way to tease mathematicians is to show them the two equal area figures pictured below and ask them where the missing box comes from. Explain to these smug tricksters why this doesn't break mathematics.

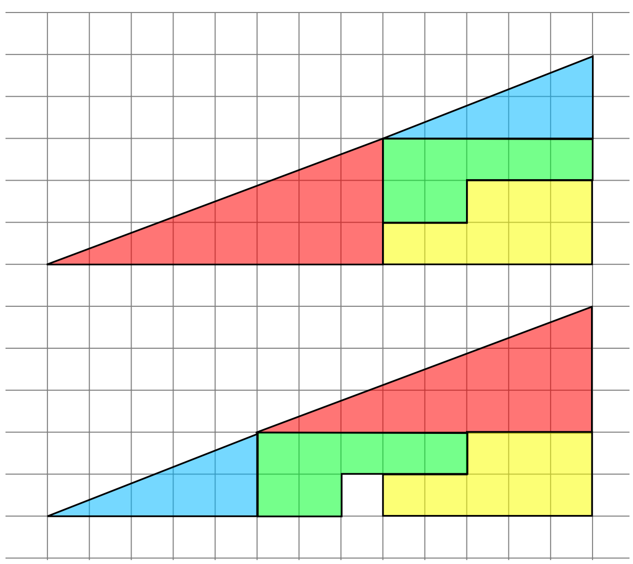

Figure 1: Two equal area figures that don't look equal area.

[1]Some problems provided by David Treeby, others from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

4. Let $ABCDEFGH$ be a cube of side 2.

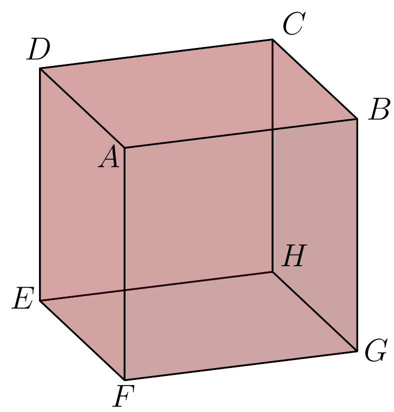

Figure 2: A cube

(a) Let M be the midpoint of BC and N the midpoint of EF. Find the area of $AMHN$.

(b) Let P be the midpoint of AB, and Q the midpoint of HE. Let AM meet CP at X, and let HN meet FQ at Y. Find the length of XY.

5. You and a friend are playing poker and you are lucky enough to obtain a four of a kind. Knowing this, does this increase or decrease the chance of your opponent obtaining a four of a kind?

6. Suppose $a_1, a_2, a_3, \ldots$ form a sequence. We can write the product

$$a_1 \times a_2 \times \cdots \times a_N = \prod_{k=1}^{N} a_k.$$

(a) Show that

$$\prod_{k=2}^{N} \frac{k-1}{k+1} = \frac{2}{N(N+1)}.$$

(b) Show also that

$$\prod_{k=2}^{N} \frac{k^2+k+1}{k^2-k+1} = \frac{N^2+N+1}{3}$$

(c) The infinitely long product

$$\prod_{k=2}^{\infty} \frac{k^3-1}{k^3+1}$$

tends to a rational number $\frac{p}{q}$ where p and q are coprime integers (that is, $\frac{p}{q}$ cannot be further simplified). Find $p+q$.

Senior Questions Consider the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}.$$

1. Show that $f(x)$ is continuous at $x = 0$.
2. Show that $f(x)$ is differentiable at $x = 0$.
3. Show that $f'(x)$ is not continuous at $x = 0$, i.e. f is not continuously differentiable.