

MATHEMATICS ENRICHMENT CLUB.

Problem Sheet 5, June 3, 2014¹

1. In the equation

$$29 + 38 + 10 + 4 + 5 + 6 + 7 = 99,$$

the left hand side contains each digit exactly once. Either find a similar expression using all the digits 0-9 and only + signs to obtain 100 or prove that it isn't possible.

2. Show that the fraction

$$\frac{21n + 4}{14n + 3}$$

cannot be simplified further for any positive integer n .

3. A point P lies inside a triangle ABC . Three lines are drawn through P parallel to the sides of ABC dividing the triangle into 6 regions, 3 of which are triangles. If the area of these smaller triangles are 12, 27 and 75 square centimetres, find the area of ABC .
4. A bakery sells donuts in packs of 5, 9 or 13. What is the largest number of donuts that cannot be bought exactly?
5. Let a_n be the Fibonacci sequence, i.e.

$$a_1 = 1, \quad a_2 = 1, \quad a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 3.$$

Prove the following:

- (a) $a_n^2 - a_{n-1}a_{n+1} = (-1)^{n-1}$
- (b) a_n is even if and only if n is a multiple of 3.
- (c) $a_n = a_{r+1}a_{n-r} + a_r a_{n-r-1}$ for $1 \leq r \leq n-2$
- (d) a_k is a factor of a_n if and only if k is a factor of n .

¹Some problems from UNSW's publication *Parabola*

Senior Questions A $n \times n$ square matrix is table of numbers that has n rows and n columns. We can write the entry in the i th row and j th column of a matrix A as $[A]_{ij}$. So

$$A = \begin{pmatrix} [A]_{11} & [A]_{12} & \cdots & & [A]_{1n} \\ [A]_{21} & [A]_{22} & [A]_{23} & \cdots & [A]_{2n} \\ [A]_{31} & & \ddots & & \\ \vdots & & & & \\ [A]_{n1} & \cdots & & & [A]_{nn} \end{pmatrix}$$

Just like numbers, square matrices of the same size can be added and multiplied together. Two matrices are equal if all of their entries are equal.

- To add square matrices of the same size together, we simply add their corresponding entries. That is $[A + B]_{ij} = [A]_{ij} + [B]_{ij}$. Prove that square matrix addition is commutative and associative, i.e.
 - $A + B = B + A$ (commutative) and
 - $A + (B + C) = (A + B) + C$ (associative).
- To multiply square matrices together we follow the rule

$$[AB]_{ij} = \sum_{k=1}^n [A]_{ik}[B]_{kj}.$$

Show that matrix multiplication

- is associative, i.e. $A(BC) = (AB)C$, but
 - is not commutative, i.e. $AB \neq BA$ for all $n \times n$ square matrices A and B .
- With real numbers we have a special number, 1, which if you multiply any number, $x \neq 0$, to it you get the same number x back, i.e. $x1 = 1x = x$. There is a similar matrix I which has the rule that for any matrix A such that not every entry is 0 (there's at least one entry $[A]_{ij} \neq 0$) we have $AI = IA = A$. Find the matrix I .