UNSW AUSTRALIA

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MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 6, June 10, 2014[1]

1. $1^3 + 2^3 + 3^3 = 36$ which is divisible by 18. Find all triplets of consecutive natural numbers such that the sum of their cubes is divisible by 18.

2. Monty Hall is the presenter of a television game show. In his game he has three doors, behind one door is the grand prize - a brand new car - behind the other two await goats, ready to bleat in mockery at the contestant's loss. Monty asks the contestant to pick a door, after which he opens one of the other two doors to reveal a goat. Now Monty, having shown the contestant a goat behind the door he opened, asks whether the contestant wishes to keep their selected door, or swap. Is it better for them to stay, swap or does it make no differene?

3. Sixteen counters lie in a 4×4 grid. Remove 6 counters so that each row and column has exactly 2 or 4 counters. (Bonus points: how many different solutions are there? Bonus bonus points: how many different solutions are there if two solutions are considered the same if you can rotate the grid to get from one solution to the other?)

4. At a party of 11 people, everyone claims they shook hands with exactly 5 other people. Show that someone is lying.

5. Show that the number of people who have an odd number of friends on Facebook must be even.

6. Let AB be a line segment whose midpoint, M, is marked. Let P be a point not on AB, show that you can, using only a straight edge and a pencil, construct a line through P, parallel to AB.

7. Let c_n be the nth term of a sequnce with $c_1 = 1$, $c_2 = -1$, and $c_n = -c_{n-1} - 2c_{n-2}$ for $n \geq 3$. Show that
$$2^{n+1} - 7c_{n-1}^2$$
is a perfect square for every integer $n \geq 2$.

[1] Some problems from UNSW's publication *Parabola*

Senior Questions

Consider an equilateral triangular hole, and the piece that fits into it. The *symmetry group* of an equilateral triangle is made up of the operations you can do to the piece so that it still fits in its hole. For instance, you can rotate it by $60°$.

1. There are 6 operations in total, list them.
2. By labelling the corners of the triangle show that these operations don't necessarily commute, that is if you do operation 1 first then operation 2 that it's not necessarily the same as doing operation 2 first and then 1.
3. As with matrices last week, there is an operation, e, called the identity, such that if x is any other operation $ex = xe = x$. Which operation does e correspond to?
4. Each operation has an "undo" operation called its inverse, such that if x is an operation and e is the identity operation, there's a y such that $xy = yx = e$. For each of the operations in the symmetry group of an equilateral triangle, list their inverse. Show also that for operation x, its inverse is unique, i.e. there's only one y such that $xy = yx = e$.