MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 1, May 6, 2014

1. Consider, for instance $2(2x+3y)+(7x+5y) = 11x+11y = 11(x+y)$, which is clearly a multiple of 11. Suppose now that $(2x+3y)$ is a multiple of 11, then so is $2(2x+3y)$, and if we wish to add a number to this and still get a multiple of 11, we must add a multiple of 11. So $7x+5y$ must also be a multiple of 11.

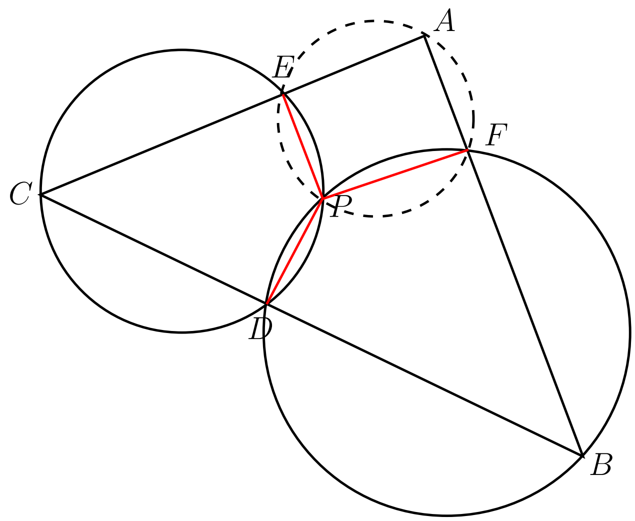
2. The number of ways to paints 6 things with 6 colours is 6!. However, a cube may be rotated, so we must divide out the number of orientations a cube has to ensure we are not counting the same colouring twice. Let's fix a face as the "top" face – we can then rotate the cube 4 times, keeping the "top" still. A cube has 6 possible faces to choose as the "top", so there are $6 \times 4 = 24$ orientations of the cube. Thus the number of colourings is $6!/24 = 30$.

3. (a) $0.75_{10} = 0.11_2$ ($0.11_2 = 1 \times \frac{1}{2^1} + 1 \times \frac{1}{2^4} = \frac{1}{2} + \frac{1}{4}$).
   (b) $0.96875_{10} = 0.11111_2$
   (c) Noticing that $0.\underbrace{111\cdots 1}_{n \text{ 1's}} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}$ we see that the infinitely long sum

   $$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^k}+\cdots = 0.11111\ldots_2 = 0.\dot{1}_2 = 1$$

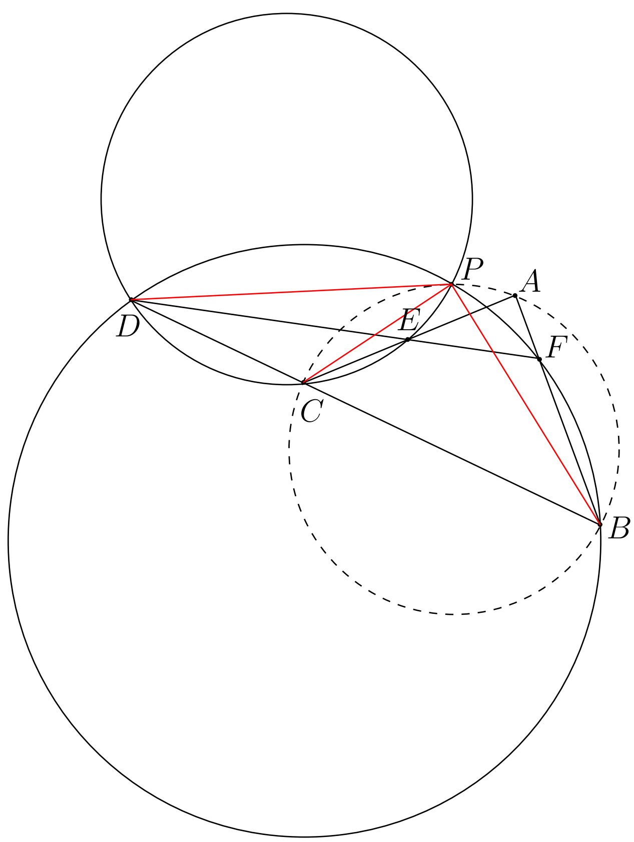
   To see this last equality, let $x = 0.\dot{1}_2$, then $2x = 1.\dot{1}_2$, so $2x - x = x = 1.\dot{1}_2 - 0.\dot{1}_2 = 1_2 = 1_{10}$.

4. The angle bisectors of any triangle all meet at the centre of the inscribed circle. So if we draw the angle bisector from the bottom left corner, it meets the elder wand at the centre of the resurrection stone. Since the cloak of invisibility is an equilateral triangle, this gives us a triangle, made of the angle bisector, the radius of the resurrection stone and half the side length of the cloak of invisiblity, with angles 30°, 60° and 90°. So the ratio of the radius of the resurrection stone to half the side length of the cloak of invisibility is $1 : \sqrt{3}$, meaning the sought after ratio is $1 : 2\sqrt{3}$.



5. (a) Draw in the red lines DP, FP and EP. Note that the quadrilaterals $EPDC$ and $FPDB$ are cyclic. So $\angle DPF = \pi - \angle DBF$ and $\angle EPD = \pi - \angle ECD$. Since the three red lines all meet at a point $\angle EPF + \angle EPD + \angle DPF = 2\pi$ so $2\pi + \angle EPF = 2\pi + \angle ECD + \angle FBD$ so $\angle EPF = \angle ECD + \angle FBD = \pi - \angle EAF$ (angle sum of triangle ABC) and so $AEPF$ is a cyclic quadrilateral so a circle must pass through those four points.

   (b) This solution provided by Michelle Royters.



Draw in the red lines DP, PB and PC. We will show that $\angle CPB = \angle CAB$

which proves that $CPAB$ is a circle. Let $\angle DPC = x$ and $\angle CPB = y$. Since angles $\angle DPC$ and $\angle DEC$ stand on the same arc of circle $DPEC$ they are equal, so $\angle DEC = x$. Similarly angles $\angle DPB$ and $\angle DFB$ stand on the same arc of circle $DPFB$ and so are equal, i.e. $\angle DFB = x + y$. Now $\angle DFB$ is exterior to triangle AEF, so $\angle DFB = x + y = \angle AEF + \angle EAF$ and $\angle AEF = x$ as it is vertically opposite $\angle DEC = x$. Thus $\angle EAF = \angle CAB = y = \angle CPB$.

6. A two digit narcisstic number with digits ab must satisfy

$$a^2 + b^2 = 10a + b$$

or

$$a^2 - 10a = b - b^2.$$

Now given an a, we can compute what b must be. For instance, for $a = 1$ we must have $b^2 - b - 9 = 0$ or $b = \frac{1}{2} \pm \frac{1}{2}\sqrt{37}$. Neither of those are the integers $0, 1, \ldots, 9$ so there are no narcissistic numbers of the form $1b$. If $a = 2$ we must have $b^2 - b - 16$ or $b = \frac{1}{2} \pm \frac{1}{2}\sqrt{65}$ which are, again, not integers, so there are no narcissistic numbers of the form $2b$. Repeat for $a = 3, 4, \ldots, 9$ and you'll see that there are no 2-digit narcissistic numbers.

Senior Questions

1. The trick here is to notice, that at some point $x = N$, the gradient of $f(x)$ is positive for all $x > N$, so no matter the value of $f(N)$, since it will always be pointing upwards, it will eventually be positive. By differentiating

$$f'(x) = \ln \frac{10}{9} - \frac{1}{x}.$$

Since $\frac{1}{x} \to 0$ and $\ln \frac{10}{9} > 0$ there's an M at which $\ln \frac{10}{9} > \frac{1}{x}$ for all $x > M$. A quick rearranging says that for $M = \frac{1}{\ln \frac{10}{9}}$, this is true. (Note that $N \neq M$, only the existence of such an M proves the existence of such an N).

2. We have shown that for $x > N$ for some N that

$$(x-1)\ln 10 - x \ln 9 - \ln x > 0$$

which, with some rearranging becomes

$$\begin{aligned} x\ln 9 + \ln x &< (x-1)\ln 10 \\ x9^x &< 10^{x-1}. \end{aligned}$$

3. Consider an n-digit narcissistic number $d_1 d_2 \cdots d_n$. It would need to have

$$d_1^n + d_2^n + \cdots + d_n^n = 10^n d_1 + 10^{n-1} d_2 + \cdots 10 d_{n-1} + d_n.$$

The largest each digit can be is 9 so the largest possible value for the left hand side in the above is $n9^n$. The smallest the right hand side may be is when $d_1 = 1$ and all other $d_j = 0$, so 10^n. But we have already shown that for large enough n, $n9^n$ is not even as great as 10^{n-1} and so can't possibly be larger than 10^n, meaning the largest possible value of the left hand side above is smaller than the smallest possible value of the right hand side, and so one could not possible equal the other, for large enough n. This all amounts to there being a largest narcissistic number, above which no number can be narcissistic. Since there are only finitely many positive integers smaller than this largest narcissistic number, there can be only finitely many narcissistic numbers in total.