MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 4, May 27, 2014[1]

1. Suppose we have a pizza of radius r that fits perfectly into a square box. The box's sides must then be $2r$, so the ratio of pizza area to box area is $\pi r^2/(4r^2) = \pi/4 \approx 0.79$. If we have a square pizza with side length x that fits neatly into a circular box, the box would have to have a diameter of $x\sqrt{2}$. So the ratio of pizza area to box area would be $x^2/\left(\frac{x^2}{2}\pi\right) = 2/\pi \approx 0.64$. Thus the more economical use of space (where the pizza fills more of the box) is the first option.

2. Let a_n be the number of ways Bec can ascend a flight of n stairs. She can ascend the first two steps either one at a time, or both at once. If she takes them one a time she can ascend the remaining $n-1$ steps in a_{n-1} ways. If she takes them both at once she can ascend the remaining $n-2$ steps in a_{n-2} ways. Since these are her only two options for the first two steps, then $a_n = a_{n-1} + a_{n-2}$, and we have a recurring formula for ascending n stairs (which may look familiar). So $a_1 = 1$ and $a_2 = 2$, so $a_{10} = 89$.

3. The trick here is that the original figure is not actually a triangle. Observe the gradient of hypotenuses of the red and blue triangles – the larger, red trangle has a hypotenuse with a gradient of 3/8 while the smaller, blue triangle has a hypotenuse with a gradient of 2/5. Since $3/8 \neq 2/5$ these two hypotenuses are not parallel, and so do not form a straight line when placed end to end.

   In fact, if you draw the top figure over the top of the bottom figure, the lines masquerading as the original figure's hypotenuse form a parrallelogram with base $b = \sqrt{3^2+8^2}$ and perpendicular height $h = \sqrt{2^2+5^2}\sin(\tan^{-1}2/5 - \tan^{-1}3/8)$. Let's call $\tan^{-1}2/5 = \theta$ and $\tan^{-1}3/8 = \phi$, then using the double angle formula we get

   $$\sin(\theta-\phi) = \sin\theta\cos\phi - \sin\phi\cos\theta.$$

   Now $\sin\theta = \sin\tan^{-1}2/5 = \frac{2}{\sqrt{2^2+5^2}}$, $\cos\theta = \frac{5}{\sqrt{2^2+5^2}}$, $\sin\phi = \frac{3}{\sqrt{3^2+8^2}}$ and $\cos\phi = \frac{8}{\sqrt{3^2+8^2}}$. So the area of the parallelogram is

   $$\sqrt{3^2+8^2}\sqrt{2^2+5^2}\left(\frac{2}{\sqrt{2^2+5^2}}\frac{8}{\sqrt{3^2+8^2}} - \frac{3}{\sqrt{3^2+8^2}}\frac{5}{\sqrt{2^2+5^2}}\right) = 1$$

   the exact size of the supposed missing square!

[1] Some problems provided by David Treeby, others from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

4. Let $ABCDEFGH$ be a cube of side 2.

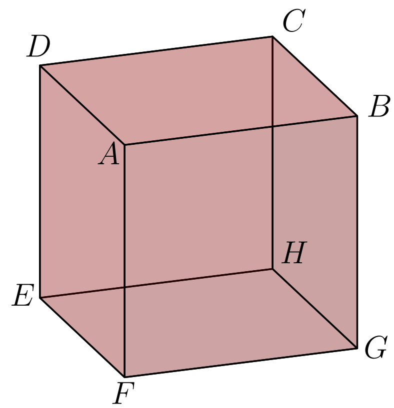

Figure 1: A cube

(a) It can be difficult to visualise what shape $AMHN$ form, or indeed if they even lie on the same plane. However, notice that AM and HN are parallel (in 3 dimensions this means they are parallel lines in parallel planes – so if we move AM down to the plane $EHGF$ it would lie parallel to HN). Similarly so are MH and AN, so we do indeed have a planar parallelogram (two pairs of parallel sides). In fact, each side is the same length so we have a rhombus. The area of a rhombus is half the product of the diagonals, which are $AH = \sqrt{2^2 + (\sqrt{2^2 + 2^2})^2} = 2\sqrt{3}$ and $NM = \sqrt{2^2 + 2^2} = 2\sqrt{2}$. So the area is $4\sqrt{6}$.

(b) Let's just look at the top of the cube, in particular triangle ABC. The lines AM and CP are medians of ABC and so divide the third median into the ratio $1 : 2$ (a proof we might have done before). The line BX extended would actually bisect AC at X' (perhaps another proof to do?). The line XX' then has length $\sqrt{2}/3$ – since BD has length $2\sqrt{2}$ so $BX' = \sqrt{2}$ and XX' is one third of BX'.

The bottom of the cube is the same picture but flipped along the diagonal of the face. So XY forms the hypotenuse to a right angled triangle whose base is $2\sqrt{2}/3$ and height 2, meaning $XY = \sqrt{2^2 + 3\frac{2^2}{3^2}}$.

5. This question may be worded a bit ambiguously, but here's how we'll interpret it. Ten cards are dealt to the two players and sit, face down on the table. What is the probability that your opponent has a 4 of a kind here? I now look at my hand and see that I have a 4 of a kind, did the probability my opponent has a 4 of kind go up or down?

For those familiar with some probability notation, let's let the B signify the event in which my opponent has a 4 of a kind, and A the event in which I have a four of kind. The question asks, is $P(B|A) \geq P(B)$? Which is read as "is the probability of B given A greater or equal to the probability of B?

The probability of B is given by

$$P(B) = \frac{C(13,1)C(4,4)C(48,1)}{C(52,5)}$$

where $C(n, m)$ is the number of ways of choosing m things from n, so here we choose 1 of the thirteen numbers, then choose all 4 of those 4 cards and for our final card choose 1 from the remaining 48. The total number of hands is the number of ways of choosing 5 cards from 52.

We can count $P(B|A)$ a number of different ways, but I'll use the formula

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

where $P(B \cap A)$ means the probability of both B and A occuring. To get both players to have a 4 of a kind, we compute

$$P(B \cap A) = \frac{C(13,1)C(4,4)C(48,1)C(11,1)C(4,4)C(43,1)}{C(52,5)C(47,5)}.$$

That is, from the 13 numbers choose 1, then choose 4 of those 4 cards, and 1 from the remaining 48 for the first player's hand. Then for the second player, choose 1 of the 11 remaining numbers (remember the fifth card in player 1's hand can't be used to form a 4 of a kind), choose 4 of those 4 cards and then 1 from the remaining 43. The total number of hands is to first choose 5 from 52 for player 1 and then 5 from the remaining 47 for player 2.

The probability, $P(A)$ is the same as $P(B)$. So

$$\begin{aligned} P(B|A) &= \frac{\frac{C(13,1)C(4,4)C(48,1)C(11,1)C(4,4)C(43,1)}{C(52,5)C(47,5)}}{\frac{C(13,1)C(4,4)C(48,1)}{C(52,5)}} \\ &= \frac{11 \times 43}{C(47,5)} = \frac{1}{3243}, \text{ and} \\ P(B) &= \frac{13 \times 38}{C(52,5)} = \frac{1}{4165}. \end{aligned}$$

So by looking at our hand and seeing we have a 4 of kind it is then more likely that our opponent has a 4 of a kind than it was before we looked.

6. Because I had too many typos in this question I'll leave its solution until next week, in case you want to have another go.

Senior Questions Consider the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}.$$

1. Here all we do is check

   $$\lim_{x \to 0} f(x) = f(0).$$

   The value of f at $x = 0$ is given by the bottom branch so $f(0) = 0$. For all $x \neq 0$, $-1 \leq \sin \frac{1}{x} \leq 1$, so $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

2. Here we must check that

   $$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

   exists. That is

   $$\lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h}$$

   exists. Again, since $\sin \frac{1}{h}$ is bounded between -1 and 1 for all $h \neq 0$, this limit equals 0 (and so exists).

3. The function $x^2 \sin \frac{1}{x}$ is differentiable everywhere, so f is too (as we have just shown). It's derivative for $x = 0$ is 0, so

   $$f'(x) = \begin{cases} 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \left(-\frac{1}{x^2}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

   Now let's check whether $\lim_{x \to 0} f'(x) = f'(0) = 0$. The limit

   $$\lim_{x \to 0} 2x \sin \frac{1}{x}$$

   exists by the same arguments above, but

   $$\lim_{x \to 0} x^2 \cos \frac{1}{x} \left(-\frac{1}{x^2}\right) = \lim_{x \to 0} - \cos \frac{1}{x}$$

   which does not tend to a specific value, and so doesn't exist. Thus the whole $\lim_{x \to 0} f'(x)$ doesn't exist, let alone equal 0. So $f'(x)$ is not continuous.

   Think about what this means – f continuously changes as x moves, there are no sudden jumps, and f even has a gradient everywhere, but this gradient doesn't vary smoothly at $x = 0$. This is an example of a function that is differentiable but not continuously differentiable (it's derivative exists but is not continuous).