MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 11, July 28, 2015

1. Alice and Carla are playing a dice game. Here’s how it works:
   • Each person rolls a die, and the highest number rolled of the two is recorded.
   • If the highest number rolled is a 1, 2, 3 or 4, Alice wins.
   • If the highest number rolled is a 5 or a 6, Carla wins.

   On average, who is more likely to win: Alice, Carla, or are the probabilities equal?

2. Find the remainder when $x^{1999}$ is divided by $x^2 - 1$.

3. How many 3 digit positive integer is/are the sum of exactly 9 distinct powers of 2?

4. Given that $a + b = 1$ and $a^2 + b^2 = 2$, what is the value of $a^7 + b^7$?

5. Let $\triangle ABC$ be right-angled. Let $A'$ be the mirror image of the point $A$ in the side $BC$, let $B'$ be the mirror image of $B$ in $AC$ and $C'$ the mirror image of $B$ in $AB$; see above. Find the ratio

   $$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle A'B'C')}$$.

6. Find all positive integers $n$ for which all of the numbers

   $$n, 2n - 1, 2n + 5, 3n - 2, 5n - 4, 6n - 5, \text{ and } 12n + 5$$

   are prime. (Note the integer 1 is not prime).

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1Some problems from UNSW’s publication Parabola, and the Tournament of Towns in Toronto.
Senior Questions

1. Find all solutions of $2^x + 3^x + 6^x = x^2$.

2. Let $f(x) = x + \int_0^1 (xy^2 + x^2 y)f(y) \, dy$. Find the value of $f(10)$.

3. Denote by $[a, b]$ the least common multiple of $a$ and $b$. Let $n$ be a positive integer such that

$$[n, n + 1] > [n, n + 2] > \ldots > [n, n + 35].$$

Prove that $[n, n + 35] > [n, n + 36]$. 