MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 10, July 21, 2015[1]

1. Given a n-digits long number, if we fix the last digit (say let it be 1), then there are $(n-1)!$ ways to arrange the rest of the $n-1$ digits (say $2, 3, \ldots, n$) to get a different n-digits long number. Hence, each of $1, 2, \ldots, n$ will appear $(n-1)!$ times in the last digit of the n-digits long number. Therefore the last digit of all combinations of n-digit long numbers contributes to the sum by an amount of $(n-1)! \times (1+2+3+\ldots+n)$.

   We can make a similar argument by fixing the second last digit, so that the second last digit of all combinations of n-digits long numbers contributes to the sum by an amount of $(n-1)! \times (1+2+3+\ldots+n) \times 10$.

   Repeating this for all digits, then the total sum is $(n-1)! \times (1+2+3+\ldots+n) \times (1+10+10^2+\ldots+10^n)$.

2. Note that $\sqrt[4]{2} \times \sqrt[8]{4} \times \sqrt[16]{8} \times \sqrt[32]{16} \times \sqrt[64]{32} \ldots = \sqrt[4]{2} \times \sqrt[8]{2^2} \times \sqrt[16]{2^3} \times \sqrt[32]{2^4} \times \sqrt[64]{2^5} \ldots$. Hence we can rewrite this product as

$$2^{\frac{1}{4}} \times 2^{\frac{2}{8}} \times 2^{\frac{3}{16}} \times 2^{\frac{4}{32}} \times 2^{\frac{5}{64}} \ldots = 2^{\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+\frac{5}{64}} \ldots.$$

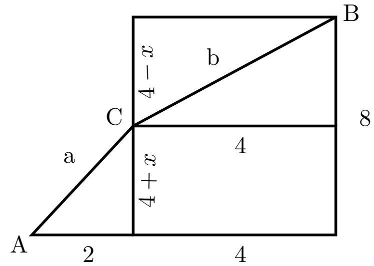
   Which means that all we need to find is the infinite sum $x = 1/4 + 2/8 + 3/16 + 4/32 + 5/64 + \ldots$. The trick here is to compare x with $x/2$:

$$\begin{aligned} x - \frac{x}{2} &= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots \\ \frac{x}{2} &= \frac{1/4}{1-1/2} \\ x &= 1. \end{aligned}$$

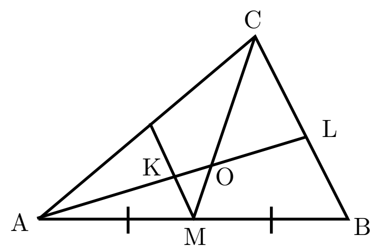
   We conclude that product of the surds is 2.

3. Let a be the greatest number written on a blackboard, pick another integer b on the board, then $a \geq b$. Furthermore, there is an integer $n \geq 0$ such that $2^n \leq a < 2^{n+1}$, so that $2^n < a + b \leq 2a < 2^{n+2}$. Since $a+b$ must be a power of two, we must have $a + b = 2^{n+1}$. Because b is an arbitrary integer we picked and there is only one choice of n, we can conclude that $2^{n+1} - a$ is the only other integer on the board; there is only two numbers on the board.

[1]Some problems from *Tournament of Towns in Toronto*.



4. Consider the above diagram. By Pythagoras, $a = \sqrt{(4+x)^2 + 4}$ and $b = \sqrt{(x-4)^2 + 16}$; the two roads that connects the point A to B has length a and b. The total length of the roads; that is $a + b$ is minimised when the two lines AC and CB in the diagram are co-linear. Therefore, the combine length of the two roads is $\sqrt{(2+4)^2 + 8^2} = 10$.



5. (a) Since the lines KM and CB are parallel, the triangles $\triangle KMO$ and $\triangle OLC$ are similar. In particular, by angles and ratios we have the formula

$$\frac{|KO|}{|KL| - |KO|} = \frac{|OM|}{|MC| - |OM|}.$$

   (b) Since M is the midpoint of AB and the line KM, CB are parallel, by the midpoint theorem K is the midpoint of AL. Additionally, by using the fact that $2|MC| = |AL|$, we have $|KL| = |MC|$. Now substituting $|KL| = |MC|$ into the formula from part (a), we obtain $|KO| = |OM$. Therefore the triangles $\triangle KMO$ and $\triangle OLC$ are isosceles. Finally, using the condition $\angle OLC = 45°$ we have $\angle COL = 90°$.

6. Since the constant coefficient of $p(x)$ is 3, $abcd = 3$. Therefore,

$$\frac{abc}{d} = \frac{3}{d^2}, \qquad \frac{acd}{b} = \frac{3}{b^2}, \qquad \frac{abd}{c} = \frac{3}{c^2}, \qquad \frac{bcd}{a} = \frac{3}{a^2}.$$

Let $y = 3/x^2$, then $p(\sqrt{3/y}) = 0$ when $p(x) = 0$. Therefore rearranging $p(\sqrt{3/y}) = 0$ gives a polynomial of y with the required roots.