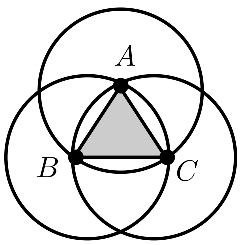
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 15, August 25, 2015[1]

1. How many distinct prime factors does $2^{32} + 2^{17} + 1$ have?

2. Find the last digit of $1^5 + 2^5 + \ldots + 123^5$.



3. Let A, B and C be the centre of the three circles shown above. The points B and C forms two arcs, with a length difference of $8\pi/3$ around the circle with centre A. Find the area of the triangle $\triangle ABC$.

4. Let $a > 1$ be a positive integer. We obtain the number b by gluing two copies of the digits of a together in order; for example, if $a = 123$ then $b = 123123$. If b is a multiple of a^2, then find all possible values of b/a^2.

5. A 10×12 rectangular paper is folded along the grid lines several times, forming a thick 1×1 square. How many pieces of paper can we possibly get by cutting the square along the segment connecting
   (a) the midpoints of a pair of opposite sides;
   (b) the midpoints of a pair of adjacent sides?

6. Consider an arbitrary number $a > 0$. We know that the inequality $10 < a^x < 100$ has exactly 5 positive integer solutions. How many solutions in positive integers may the inequality $100 < a^x < 1000$ have? In each case, list the solutions.

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*.

Senior Questions

1. Seventeen primes $p_1 < p_2 < \ldots < p_{17}$ have the property that the sum of their squares is also a square. Prove that $p_{17}^2 - p_{16}^2$ is divisible by p_1.

2. Integers $1, 2, \ldots, 100$ are written in a circle, not necessarily in that order. Can it be that the absolute value of the difference between any two adjacent integers is at least 30 and at most 50?

3. Twelve knights $k_1, k_2, \ldots, k_{12}$ are seated in anti-clockwise order around a circular table. What is the minimal number of swaps required to change their order to a clockwise one, if any swap can be made only between adjacent knights? What is the answer if there are thirteen knights?