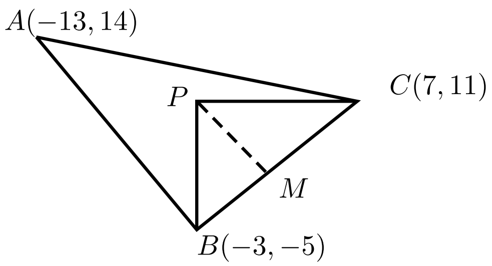
**MATHEMATICS ENRICHMENT CLUB.**
**Solution Sheet 12, August 2, 2015**

1. (a) The remainder of 2017 divided by 3 is 1, so the remainder of $2017^{46}$ divided by 3 is $1^{46} = 1$. Also, the remainder of 46 when divided by 3 is 1. Hence, the remainder of $2017^{46} - 46$ divided by 3 is zero; that is $2017^{46} - 46$ is not prime.

   (b) Apply the same technique as in part $(a)$, we can show that $2017^{46} + 46$ is divisible by 5.

2. Let $f(x) = x^3 + px^2 - x + q$. Then since $(x - 5)$ is a factor of $f(x)$, $x = 5$ is a root of $f(x)$ which means $f(5) = 125 + 25p - 5 + q = 0$. Furthermore, $f(x)$ has remainder 24 when divided by $(x - 1)$ which means $f(1) = 1 + p - 1 + q = 24$. We now have two equations of $p$ and $q$, thus we can solve simultaneously to obtain $p = 6$ and $q = 18$.




3. Let $M$ be the point of interception between the line $BC$ and the line perpendicular to $BC$ and passing through the point $P$; see above. Then since $\triangle PBC$ is equilateral, $M$ is the midpoint of $BC$ so $M$ has coordinates $(2, 3)$. To find the coordinate of $P$, we seek to determine the distance between $M$ and $P$ in both the $x$ and $y$ directions. To do this, we find the gradient and length of $PM$. The gradient of $BC$ is $8/5$, thus the gradient of $PM$ is $-5/8$. Note that $|PM| = |PB| \sin 60^\circ = |BC| \sin 60^\circ$ , hence $|PM| = \sqrt{3 \times 89}$.

   Let $x$ and $y$ be the distance in the $x$ and $y$ directions from $M$ to $P$. Then using the gradient of $PM$, we know that $x = -8/5y$, and therefore

$$|PM| = \sqrt{x^2 + y^2}$$
$$\sqrt{3 \times 89} = \sqrt{x^2 + \frac{25}{64}x^2}$$
$$x = \pm 8\sqrt{3}.$$

It is clear from the diagram that $x = -8\sqrt{3}$ else $p$ would be outside of $\triangle ABC$, hence $y = 5\sqrt{3}$. Therefore the coordinate of the point $P$ is $(2 - 8\sqrt{3}, 2 + 8\sqrt{3})$.

4. The answer is 3; The sequence is $\ldots 3, 4, 5, 6, 7, 8, 9, \ldots$, the sum of the middle five terms are equal to the sum of the last four. So to complete this problem, we just need to show that there is no way to get less than 3 overlaps.

   Firstly, the sum of five consecutive terms is 30, the sequence must contain 6. Also, the sequence cannot be constant, because if it is then the sum of four consecutive terms is 24. Therefore, the common difference $d \geq 1$.

   If the overlap is less than three terms, then the sum of the four consecutive terms cannot include 6. Additionally, the four consecutive terms being added must all be greater than 6 (otherwise the sum will be less than 30).

   By the assumption of the existence of less than three overlaps, we can write the sum of the four consecutive terms as $S_4 = [6+kd]+[6+(k+1)d]+[6+(k+2)d]+[6+(k+3)d] = 24 + 4kd + 6d$ for some $k \geq 1$. But since $d \geq 1$, we can see that there is no solution for $S_4 = 30$; we have a contradiction.

5. The trick is the add $f(\frac{n}{2015})$ and $f(\frac{2015-n}{2015})$, for each $1 \leq n \leq 2014$. Since

$$\begin{aligned} f\left(\frac{n}{2015}\right) + f\left(\frac{2015-n}{2015}\right) &= \frac{4^{\frac{n}{2015}}}{4^{\frac{n}{2015}} + 2} + \frac{4^{\frac{2015-n}{2015}}}{4^{\frac{2015-n}{2015}} + 2} \\ &= \frac{4^{\frac{n}{2015}} \times \left(4^{\frac{n}{2015}} - 2\right)}{2\left(4^{\frac{2015-n}{2015}} - 4^{\frac{n}{2015}}\right)} + \frac{4^{\frac{2015-n}{2015}} \times \left(4^{\frac{2015-n}{2015}} - 2\right)}{2\left(4^{\frac{n}{2015}} - 4^{\frac{2015-n}{2015}}\right)} \\ &= 1. \end{aligned}$$

   Hence, $f(1/2015) + f(2/2015) + \ldots + f(2014/2015) = 2014/2 = 1012$.

6. One way to solve this is by using similar triangles, but a much cleaner way to do this is by introducing coordinates. Since the diagonals of $ABCD$ meets at right angles, if we set $(0, 0)$ be the point where the diagonals intercept, then we can assign the coordinates $A(a, 0), B(0, b), C(c, 0)$ and $D(0, d)$, where $a, d < 0$ and $b, c > 0$.

   We find the $y$-coordinates $y_M$ and $y_N$ of the points $M$ and $N$ respectively, and show that they are the same; then since $A$ and $C$ are on the $x$-axis, the line $MN$ is parallel to $AC$.

   To find the $y$-coordinate of $M$, we find the point of interception between the lines $BM$ and $AD$. The equation of the line $AD$ is given by $y = (-d/a)x + d$, and the line $BM$ by $y = (-c/b)x + b$ (the line $BC$ has gradient $b/c$ which is perpendicular to $BM$). Thus solving simultaneously, we have $y_M = \frac{ac(d-b^2d)}{ac-bd}$. To find the $y$-coordinate of $N$, we just need to permute $a$ and $c$. But since $y_M$ is symmetric in $a$ and $c$, $y_M = y_N$ so we are done.

**Senior Questions**

1. Without loss of generality, suppose $\alpha \leq \beta \leq \gamma$. Since $f'(x) = 3x^2 + 2x - 5$, the turning points of $f(x)$ are $x = 1$ and $-1\frac{2}{3}$. Furthermore, $f''(x) = 6x + 2$ so $x = 1$ is a minimum and $x = -1\frac{2}{3}$ is a maximum. Therefore, the three roots satisfies $\alpha < -1\frac{2}{3}$, $-1\frac{2}{3} < \beta < 1$ and $1 < \gamma$, where we have strict inequality because the roots are not on the $x$-axis. By testing integral points between $-3$ to 2, we arrive at $-3 < \alpha < -2$, $-1 < \beta < 0$ and $1 < \gamma < 2$. Hence, $[\alpha] + [\beta] + [\gamma] = -3$.

2. Splitting the products into blocks of $m$ factors, we find that

$$\begin{aligned}
(nm)! &= (1 \times 2 \times \ldots \times m) \\
&\quad \times ((m+1) \times (m+2) \times \ldots \times 2m) \\
&\quad \times ((2m+1) \times (2m+2) \times \ldots \times 3m) \\
&\quad \times \ldots \\
&\quad \times (((n-1)m+1) \times ((n-1)m+2) \times \ldots \times nm) \\
&\geq (1 \times 2 \times \ldots \times m) \times (2 \times 4 \times \ldots \times 2m) \\
&\quad \times (3 \times 6 \times \ldots \times 3m) \times \ldots \times (n \times 2n \times \ldots \times mn) \\
&= m! \times (2^m \times m!) \times (3^m \times m!) \times \ldots \times (n^m \times m!) \\
&= \underbrace{m! \times m! \times m! \times \ldots m!}_{n \text{ lots of } m!} \times (1 \times 2 \times 3 \times \ldots \times n)^m \\
&= (m!)^n (n!)^m,
\end{aligned}$$

where we have used the fact that $m \geq 1, 2, \ldots m$ to obtain the inequality on the above equation.

3. Raise both sides of the first equation to the power $x + 4y$, and both sides of the second to the power $x + 2y$. Then we have

$$x^{(x+y)(x+4y)} = y^{(x+2y)(x+4y)} \quad \text{and} \quad x^{(2x+y)(x+2y)} = y^{(x+4y)(x+2y)},$$

and hence

$$x^{(x+y)(x+4y)} = x^{(2x+y)(x+2y)}.$$

Since $x$ and $y$ are not zero, there are now two possibilities: either $x = 1$ or the exponents on the left and right hand sides are equal. If $x = 1$ it is easy to see that $y = 1$. Otherwise,

$$(x+y)(x+4y) = (2x+y)(x+2y) \Rightarrow x^2 = 2y^2 \Rightarrow x = \sqrt{2}y.$$

Substituting back into the first of the given equations,

$$x^{(1+\sqrt{2})y} = y^{\sqrt{2}(1+\sqrt{2})y}$$

and so $x = y^{\sqrt{2}}$. Hence we have

$$y^{\sqrt{2}} = \sqrt{2}y \Rightarrow y^{\sqrt{2}-1} = \sqrt{2} \Rightarrow y = \sqrt{2}^{1/(\sqrt{2}-1)} = \sqrt{2}^{\sqrt{2}+1}$$

and $x = \sqrt{2}y = \sqrt{2}^{\sqrt{2}+2}$. So the equations have two solutions,

$$x = 1, y = 1 \quad \text{and} \quad x = \sqrt{2}^{\sqrt{2}+2}, y = \sqrt{2}^{\sqrt{2}+1}.$$