MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 13, August 11, 2015[1]

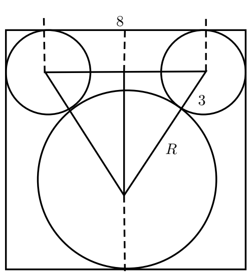
1. The prime number $p = 5$ works, because 101 and 151 are both prime numbers. We show that this is the only prime that works. First note that the numbers with remainders $1, 2, 3$ or 4 when divided by 5 covers all possible integers not divisible by 5; i.e suppose the number x_1 has remainder 1 when divided by 5, then x_1 must be one of $1, 6, 11, \ldots$.

   Now to check that $4x_1^2 + 1$ and $6x_1^2 + 1$ are not prime, we argue has follows: since x_1 has remainder 1 when divided by 5, $4x_1^2 + 1$ has remainder $4 \times 1^2 + 1 = 5$ when divided by 5. Hence $4x_1^2 + 1$ is divisible by 5 and therefore not prime. We can repeat this for the other numbers x_2 (which has remainder 2 divided by 5), x_3 and x_4, and show that it fails to be the desired prime number each time.

2. By Pythagoras we know that $c^2 \geq a^2 + b^2$. So if we assume $c^{n-1} > a^{n-1} + b^{n-1}$, then

$$c^{n-1} \times c^2 > a^{n-1} \times c^2 + b^{n-1} \times c^2$$
$$c^{n+1} \geq a^{n+1} + b^{n+1},$$

   so that the inequality holds by the standard induction arguments.



3. Let R be the radius of the big circle. Draw a triangle that connects the centre of each circles, then bisect the this triangle into two right-angled triangles; as shown above.

   From the diagram, we can see that the right-angled triangles has sides of length $R + 3$ (you may want to verify this), $(14 - 3 \times 2) \div 2 = 4$ and $14 - R - 3 = 11 - R$. Now by Pythagoras, we have

$$4^2 + (11 - R)^2 = (R + 3)^2.$$

   Solving the above equation gives $R = 128/28$.

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*.

4. First we write

$$\frac{n^2+11n+2}{n+5} = \frac{n^2+10n-3}{n+5} + 1,$$

and then we complete the square on the nominator of the fraction appearing in the RHS of the above equation,

$$\begin{aligned}\frac{n^2+10n-3}{n+5} + 1 &= \frac{(n+5)^2-28}{n+5} + 1 \\ &= n+5-\frac{28}{n+5}+1.\end{aligned}$$

Now to get an integer on the last line of the above equation, the second term tells us that 28 must be divisible by $n+5$. Since the factors of 28 are $\pm 1, \pm 2, \pm 4, \pm 7, \pm 14, \pm 28$, we conclude that $n = 2, 9$ or 23 (here we eliminated the factors that gives a negative n value).

5. The example shows that -4 is in T. We have further that -1 is in T, because $-1 = -(5-4) = -(0+1)$. Also -3 is in T, because $-3 = -(4-1) = -(0+3)$. Continuing in this way, we can eventually obtain $\{-5, -4, \ldots, 4, 5\} \in T$; that is the integers from -5 to 5 are all elements of the set T.

   Now to show that every integer is in T, we argue with induction as follows: suppose the set of integers $\{-n, \ldots, n\}$ is in T, since we already know that the case of $n = 5$ is true by the above, it remains to show that $-n-1$ and $n+1$ is also in T. $n+1$ is in T, because $n+1 = -[(-1)+(-n)] = -[(-2)+(-n+1)]$. It follows that $-n-1$ is also in T, because $-n-1 = -[n+1+0] = -[(n-1)+2]$.

Senior Questions

1. The 99 locked boxes can all be opened as long as the key to the smashed box is contained in the very last box we open.

   (a) We can think about this as placing 99 balls in a bag, with one of the ball painted red. The red ball represents the box that contains the key to the smashed box (we consider 99 boxes here because the first ball we draw is the box we smash). Hence, we want to draw the red ball last; the chance of this to occur is 1/99.

   (b) We now have two red balls in the bag, so the chance to open all locked boxes becomes are 2/99.

2. Let the centre of the unit circle be the point with coordinates $(0, 0)$, and the length of the shaded square be r. If (x, y) is the coordinate of the top left corner of the shaded square, then $y = r/2$ and $x = -\sqrt{2}/2 - r$. Furthermore, since any point that lays on the unit circle is given by the equation $x^2 + y^2 = 1$, we can find r by solving

$$(-\sqrt{2}/2 - r)^2 + (r/2)^2 = 1,$$

   which gives $r^2 = 0.08$.

3. Rewrite the equation in the form $a(c-b) = (10b+c)d$, then since a and d are co-prime, we can conclude that $(c-b)$ is positive and divisible by d. Thus $c = b + kd$, where k is a positive integer. Substituting the last equation of c back into the original equation gives $(a-d)k = 11b$. Since c and b are co-prime, the expression for c implies that b and k are also co-prime otherwise c will be divisible by the greatest common divisor of b and k. Therefore, using the fact that $(a-d)k = 11b$, either $k = 1$ or $k = 11$. It follows that we have either $c = b + d$ or $a = b + d$.

4. Arrange the stones in a circle ordered by increasing weights of the stones clockwise, then move the stickers on each stone one position counter clockwise. The results is that the heaviest stone is now labelled with the lightest weight, with all the other stones labelled with a sticker heavier than its true weight.

   Now if we are to select any number of stones from $1-99$, as long as we exclude the heaviest stone from our selection, the total true weight of the stones we selected will always be lighter than the total weight on their stickers. On the other hand, the set of stones we did not select will always have true weight under the total weight given by their stickers, because the total true weight of the stones have not change.