MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 14, August 18, 2015[1]

1. Let x be the number of cards removed. The probability to draw an ace from the reduced deck is $4/(52-x)$, the second ace $3/(52-x-1)$, the third $2/(52-x-2)$ and last $1/(52-x-3)$. Multiplying gives

$$\frac{4}{52-x} \times \frac{3}{52-x-1} \times \frac{2}{52-x-2} \times \frac{1}{52-x-3} = \frac{1}{1001}.$$

Solving for the above gives $x = 38$.

2. Since $1+2!+3! = 9 = 3^2$, we know that $n = 3$ is a possible solution to the problem. We show that $n = 3$ is the largest solution. Observe that for any number to be a perfect square, it cannot end in the digit 3. Since $1+2!+3!+4! = 33$, $n = 4$ is not a solution. Moreover, $n!$ contains the both factors 2 and 5 for $n > 4$, therefore $n!$ ends in the digit 0 for $n > 4$. We can now conclude that the number $1 + 2! + 3! + \ldots + (n-1)! + n!$ ends in the digit 3 for $n > 4$, thus cannot be a perfect square.

3. Since we are adding consecutive numbers, we know that $a_2 = a_1 + 1, a_3 = a_1 + 2, \ldots, a_{100} = a_1 + 99$. Therefore, we can write

$$\begin{aligned}\sqrt{a_2 + a_3 + \ldots + a_{99}} - \sqrt{a_1 + a_{100}} &= \sqrt{98a_1 + 1 + 2 + \ldots + 98} - \sqrt{2a_1 + 99} \\ &= \sqrt{99a_1 + 4851} - \sqrt{2a_1 + 99}.\end{aligned}$$

The second line of the equation above is minimal when $a_1 = 1$.

4. Let abc and efg be three digit numbers. Then we can write the initial 6-digits number x as

$$x = 1000 \times abc + efg.$$

Also

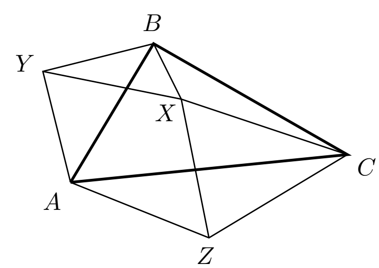
$$6x = 1000 \times efg + abc.$$

Combining the above two equations gives $5999 \times abc = 994 \times efg$, which can be further simplified to $857 \times abc = 142 \times efg$. Hence the number we are after is 142857.

[1] Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*.

5. To have all frogs the same colour, we must first reach a situation where there is a same number of frogs of two different colours. So we can think about this problem in terms of the difference between the number of frogs having two different colours, then it is possible to have all frogs the same colour if this number is zero for any combination of two colours; For example, initial this number is 1 if we compare brown with green or green with yellow, and 2 if we compare brown with yellow.

   Now in an event of two frogs with different colours meeting, both frogs change colour to the third, so the number of frogs of different colours either change by 3 or remain the same. Since we started with a difference number of 1 or 2, and can only change this number by 3, it is not possible to get a same number of frogs of two different colours.



6. See diagram above. Let $\angle BAC = a$, $\angle ABC = b$ and $\angle ACB = c$. Further, using similarity of the triangles $\triangle YBA$, $\triangle ZAC$ and $\triangle XBC$, let us denote

$$\angle YAB = \angle ZCA = \angle XCB = \alpha,$$
$$\angle YBA = \angle ZAC = \angle XBC = \beta,$$
$$\angle AYB = \angle CZA = \angle CXB = \gamma.$$

   (a) Since the $\triangle YBA$ is similar to $\triangle XBC$, we have $YB : AB = XB : BC$. It follows that $YB : BX = AB : BC$. Since $\angle YBA = \angle XBC$ we have $\angle YBX = b$. Therefore $\triangle YBX$ is similar to $\triangle ABC$.

   We can use the similarity of $\triangle ZAC$ and $\triangle XBC$ and follow the same steps as before, to show that $\triangle ZXC$ is similar to $\triangle ABC$.

   (b) The similarity between $\triangle YBX$ and $\triangle ZXC$ to $\triangle ABC$ implies $\angle XYB = a$, $\angle YXB = c$ and $\angle XZC = a$, $\angle ZXC = b$. Therefore

$$\angle AYX = \angle AYB - \angle XYB = \gamma - a,$$

   and

$$\angle AZX = \angle AZC - \angle XZC = \gamma - a.$$

   Therefore, a pair of opposite angles of the quadrilateral $AYXZ$ is equal. Further,

$$\begin{aligned}\angle YXZ = 360^\circ - \angle YXB - \angle ZXC - \angle BXC &= 360^\circ - c - b - \gamma \\ &= (180^\circ - c - b) + (180^\circ - \gamma) = a + \alpha + \beta = \angle YAZ\end{aligned}$$

   thus the other pair of angles of $AYXZ$ are also equal.

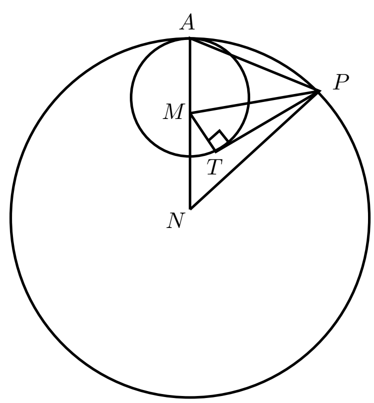
Senior Questions

1. Using the method of partial fractions, we can write

$$\frac{2n-1}{n(n+1)(n+2)} = \frac{-1}{2n} + \frac{3}{n+1} + \frac{-5}{2(n+2)}.$$

So if we let $S = \sum_{n=1}^{25} \frac{1}{n}$, then

$$\begin{aligned}\sum_{n=1}^{25} \frac{2n-1}{n(n+1)(n+2)} &= \sum_{n=1}^{25} \frac{-1}{2n} + \frac{3}{n+1} + \frac{-5}{2(n+2)} \\ &= \sum_{n=1}^{25} \frac{-1}{2n} + \sum_{n=1}^{25} \frac{3}{n+1} + \sum_{n=1}^{25} \frac{-5}{2(n+2)} \\ &= \sum_{n=1}^{25} \frac{-1}{2n} + 3\sum_{n=2}^{26} \frac{1}{n} - \frac{5}{2}\sum_{n=3}^{27} \frac{1}{n} \\ &= \frac{-1}{2}S + 3\left(-1 + S + \frac{1}{26}\right) - \frac{5}{2}\left(-1 - \frac{1}{2} + S + \frac{1}{26} + \frac{1}{27}\right) = \frac{475}{702}.\end{aligned}$$



2. Let M, N be the centres, and r, R the radii of the smaller and larger circles respectively; as shown above. Denote the angle $\angle MNP$ by θ. By the cosine rule in $\triangle ANP$ we have

$$|PA|^2 = 2R^2 - 2R^2\cos\theta = 2R^2(1-\cos\theta).$$

Similarly, the cosine rule in $\triangle MNP$ gives

$$|PM|^2 = R^2 + (R-r)^2 - 2R(R-r)\cos\theta,$$

and since $\angle MTP$ is right angle, we can apply Pythagoras on $\triangle PMT$ to obtain

$$|PT|^2 = |PM|^2 - r^2 = 2R(R-r)(1-\cos\theta).$$

Therefore, we have

$$\frac{|PT|}{|PA|} = \sqrt{\frac{R-r}{R}},$$

which is constant.

3. The answer is yes.