MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 15, August 25, 2015[1]

1. Write $2^{32} + 2^{17} + 1 = (2^{16} + 1)^2$, and observe that $2^{16} + 1$ is the largest Fermat prime number.

2. First we show that the 5^{th} power of any integer have the same last digit as the original number. For a n-digit long integer x, we can write it as $x = a_1 + 10 \times a_2 + \ldots + 10^{n-1} \times a_n$, where $a_1, a_2, \ldots, a_n$ are nonnegative integers less than 10. Then

$$x^2 = a_1^2 + 10 \times a_1 \times a_2 + 10^2 \times a_2^2;$$

   in words, the last digit of x^2 will be the same as the last digit of a_1^2. Repeating this calculation, we can show that the last digit of x^5 will be the same as the last digit of a_1^5. Now, we can easily verify that a_1^5 has the same last digit as a_1 for $a_1 = 1, 2, \ldots 9$, hence x^5 will have the same last digit as x.

   Therefore, the last digit of $1^5 + 2^5 + \ldots + 123^5$ is equal to the last digit of $1 + 2 + \ldots 123$; which is 3.

3.

4. Suppose a is n-digits long, then $b = a(10^n + 1)$. Also, $b = ka^2$ for some integer k. Therefore,

$$k = \frac{b}{a^2} = \frac{10^n + 1}{a}.$$

   Since $10^n + 1$ is a $n + 1$ digits long number, the fraction on the RHS of the last equation must be greater than 0 and less than 10. Therefore, the integer $1 < k < 9$. Since $ak = 10^n + 1$, k must be odd. k can not be 1, otherwise a will be $n + 1$ digits long. k can not be $3, 5$ or 9, because $10^n + 1$ is never divisible by those numbers. Thus the only possibility if 7.

5. (a) If we draw a horizontal line across any one of the 1×1 grid squares (we think of the dimension as *vertical* $\times$ *horizontal*), then no matter how the 10×12 paper is folded along the grid lines, this horizontal line will still be horizontal. Similarly, any vertical line will be preserved under the action of folding.

[1] Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*.

Thus, if we were to cut horizontally across the thick 1×1 square folded paper, then we are cutting horizontally across each of 1×1 grid squares of the unfold paper. This means we will produce $10 + 1$ strips of papers. Similarly, if we cut vertically then we will produce $12 + 1$ strips.

(b) Label the first row along the grid points (i.e the corner of the 1×1 squares) with A's and B's in a alternating fashion. Then label the second row with C's and D's in an alternating fashion. Repeat the AB labelling on the third row, and CD on the fourth row etc, in an alternation fashion until each grid point has been labelled. Then no matter how the 10×12 paper is folded, the A, B, C and D corners will always be folded onto itself.

Thus, if we were to cut the corner labelled by A of the thick 1×1 square folded paper, then we would be cutting out each of the grid labelled with an A in the original unfold 10×12 paper. Since there are 5×6 grid points labelled with an A, it follows that there are $30 + 1$ separated pieces due to this cut. We can use a similar argument to work the number of components we get by cutting the other corners of the folded 1×1 paper.

6. Let $a = 10^b$, then we can rewrite the inequality $10 < a^x < 100$ as $1 < bx < 2$. Similarly, if $100 < a^x < 1000$, then $2 < bx < 3$. Suppose n is the smallest integral solution to the inequality, then since there are exactly 5 solutions, the largest solution must be $n + 4$. From this we can deduce $b(n-1) < 1 < bn$ and $b(n+4) < 2 < b(n+5)$. Summing up the first inequality with itself and with the second one we obtain $b(2n-2) < 2 < b(2n)$ and $b(2n+3) < 3 < b(2n+5)$. Therefore, the inequality $2 < bx < 3$ has from 4 to 6 integer solutions; $2n, 2n+1 \ldots, 2n+4$ are always solutions, while $2n-1$ and $2n+4$ may or may not be.

So if we want to get a only four solutions, then we need to consider a number b such that $b(2n-2) < 2$ and $3 < b(2n+5)$ for some integer n. An easy way to do this is set $n = 5$, then $\frac{1}{5} < b < \frac{1}{4}$. The solutions for the first inequality is $5, 6, 7, 8$ and the second $10, 11, 12, 13$.

We can get 5 or 6 solutions by picking the appropriate b.

Senior Questions

1. Notice that we have the greatest control over the number p_1, so we want to find out what p_1 is allow to be. Suppose $p_1 > 3$, then $p_1, \ldots p_{17}$ can not contain factors of 3. Therefore, $p_i = 1 \pmod 3$ or $p_i = 2 \pmod 3$ for $i = 1, 2, \ldots, 17$; that is $p_1, \ldots, p_{17}$ must have remainder 1 or 2 when divided by 3. From this, we have $p_i^2 = 1 \pmod 3$ for each $i = 1, \ldots 17$, and so $p_1^2 + p_2^2 + \ldots + p_{17}^2 = 2 \pmod 3$. On the other hand, the square of an integer must have remainder 0 or 1 when it is divided by 3 (e.g consider remainders of the square of an even or odd number when divided by 3). Therefore, $p_1^2 + \ldots + p_{17}^2$ is not a square, so we have shown that $p_1 \leq 3$.

If $p_1 = 2$, then $p_{17}^2 - p_{16}^2$ is an even number so it is divisible by $p_1 = 2$. If $p_1 = 3$, then as before $p_{16} \equiv p_{17} = 1 \pmod 3$. Thus, $p_{17}^2 - p_{16}^2 = 0 \pmod 3$ which conclusions the proof.

2. Suppose we can place the numbers on a circle so that the condition holds. Let us call the integers from 26 to 75 *normal*, and all the others *extreme*. Two extreme integers cannot be consecutive (their difference is either less than 25 or greater than 50). Note that the numbers of the extreme and normal integers are the same and therefore they must alternate. However the normal number 26 can be adjacent to only one extreme integer 76. contradiction.

3. We consider the general case of n knights, $k_1, k_2, \ldots k_n$, where $n \geq 3$. Let $s_1, s_2, \ldots, s_n$ be the initial seats where $k_1, k_2, \ldots, k_n$ sits in order, and let $a = \lfloor n/2 \rfloor$ be the greatest integer less than or equal to $n/2$. We split the knights into the two groups $K_1 = \{k_1, k_2, \ldots, k_a\}$ and $K_2\{k_{a+1}, k_{a+2}, \ldots, k_n\}$, then we can change the anti-clockwise ordering of the seated knights into an clockwise ordering, by reversing the order of the knights in the set K_1 and K_2. To move k_1 to s_a, k_1 must swap position with k_2 then k_3 and so on sucessively; it takes $(a-1)$ swaps to move k_1 to the seat s_a. Similarly, it takes $(a-2)$ swaps to move k_2 into s_{a-1}, $(a-3)$ swaps to move k_3 to s_{a-2} and so on. Therefore, it takes $1 + 2 + \ldots + (a-1)$ swaps to reverse the order of the set K_1. Similarly, it takes $1 + 2 + \ldots + (n-a-1)$ swaps to reverse the order of the set K_2. In summery, then number of swaps required is

$$[1 + 2 + \ldots + (a-1)] + [1 + 2 + \ldots + (n-a-1)] = \sum_{r=2}^{n-1} \left\lfloor \frac{r}{2} \right\rfloor$$

Therefore, if $n = 12$ then the number of swaps required is 30, and if $n = 13$ then the number of swaps required is 36. All is left to do is to show that the number a we picked initial does indeed produce the minimum number of required swaps.