MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 16, September 1, 2015[1]

1. Suppose we take four socks and non of them match, then the next sock we draw must form a pair with one of the socks we have taken. Now to get another pair of socks, in the worst case we have to take two more socks. Therefore we have to take at least $4 + 2 \times 5$ socks from the draw to guarantee that five of the pairs are matching.

2. For a number to be divisible by 12 it must also be divisible by any factors of 12; that is $2, 3, 4, 6$. Recall that for a number to be divisible by 3, then the sum of it digits must be divisible by 3. For a number to be divisible by 4, the last two digit must be divisible by 4. For a number to be divisible by 4 then it must be even. Finall, since 2 and 3 are co-prime if a number is divisible by 2 and 3 then it must be divisible by $2 \times 3 = 6$.

   So this 5-digit number must be divisible by both 3 and 4. Since the sum of the digits of the 5-digits numbers we are forming is always 15, we need not check the divisibility by 3. Now for this 5-digits number to be divisible by by 4, then the possibility for the last two digits are $12, 24, 32$ and 52. For each choice of the list last two digits, there is $3!$ ways to rearrange the starting three digits, hence there are $3! \times 4 = 24$ ways to form

3. Label the wrestler from 1 to 100 according to their strength in increasing order. In the first round, each wrestler labelled with an even number fights a wrestler with a label one less than theirs. Then all wrestlers labelled with an odd number are losers. In the second round, each odd number wrestler fights a wrestlers with a label one less than theirs with the exception of 1 and 100; they will fight each other. Then all even wrestlers except 100 are losers. Hence only strongest wrestler takes a prize.

4. Since $33! = 33 \times 32 \times \ldots \times 2 \times 1$, we just need to count (allowing repeats) the number of integers less than 33 that contains a factor of $2, 2^2, 2^3, 2^4$ and 2^5. The answer is 31.

5. Observe that $y = 0, x = 1, p = 2$ and $y = 1, x = 1, p = 5$ are two possible solutions. We show that there is no other. Assume $y > 0$, and write $p^x = y^4+4 = y^4+4y^2-4y^2+4 = (y^2 - 2y + 2)(y^2 + 2y + 2)$. Then $y^2 - 2y + 2 < y^2 + 2y + 2$, which implies $y^2 + 2y + 2$ is divisible by $y^2 - 2y + 2$ because their product is a power of the prime number p. On the other hand, we have $y^2 + 2y + 2 = (y^2 - 2y + 2) + 4y$; i.e $y^2 + 2y + 2$ has remainder $4y$ when it is divided by $y^2 - 2y + 2$. But since $y^2 + 2y + 2$ is divisible by $y^2 - 2y + 2$,

[1] Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*.

we can conclude that $4y$ is either zero or $4y \geq y^2 - 2y + 2$. The former is impossible because $y > 0$ by assumption, and the latter implies $1 \leq y \leq 5$. We can easily check that there is no solution to the $p^x = y^4 + 4$ when $y = 2, 3, 4$ or 5.

6. Arrange the weights in increasing order, $a_1 < a_2 < \ldots < a_{11}$. Since the difference between any two weights is at least 1, we have $a_n \geq a_m + (n - m)$ for $m < n$. This implies,

   $a_7 + a_8 + a_9 + a_{10} + a_{11} \geq (a_2 + 5) +) a_3 + 5) + \ldots + (a_6 + 5) = a_2 + a_3 + a_4 + a_5 + a_6 + 25.$

   Additionally, according to the given we have

   $$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 > a_7 + a_8 + a_9 + a_{10} + a_{11}.$$

   Combining the above two displayed equations, we conclude that $a_1 > 25$, but then $a_{11} = a_1 + 10 > 35$.

Senior Questions

1. All even integers n and $n = 1$.

   The statement can be reformulated as "for any polynomial $R(x)$ with $\deg R \leq n$, does there exist mnonmials ax^k and bx^l with $0 \leq k < l \leq n$ such that $R(x) + ax^k + bx^l$ has no zeros."

   Let $n \geq 3$ be odd. Then for $R(x) = xn^+x$ we must pick up $bx^l = -x^n$ (otherwise we have a polynomial of odd degree which has a 0). We have $R(x) - x^n = x$ and if $k \geq 1$ then $R(x) + bx^l + ax^k$ is 0 as $x = 0$; if $k = 0$ we have $x + a$ which has 0 as well.

   For $n = 1$ we can always make $R(x) + bx + a - 1$ which does not vanish.

   Let n be even. Then add a monomial of degree n to make the leading coefficient positive. The resulting polynomial is bounded from below by some M. Adding the constant $1 - M$, we make the minimal value equal to 1 and get the required polynomial.

2.

3. Let the numbers be a, b, c, d and e increasing order.

   (a) Suppose the it is possible to form a triangle with sides length equal to these numbers, then $a + b > e$. Hence

   $$a^2 + b^2 + c^2 + d^2 + e^2 < ab + bc + cd + de + (a + b)e,$$

   which is a contradiction.

   (b) Let us consider the following cases:

      i. $b + c \leq d$. Then each of six triples in which two numbers are from the set $\{a, b, c\}$ and the third number is from the set $\{d, e, \}$ does not form a triangle.

      ii. $c + d \leq e$. Then each of six triples which includes e does not form a triangle.

iii. $b + d \leq e$ and $a + b \leq d$. Then each of six triples $\{a, b, d\}$, $\{a, b, e\}$, $\{a, c, e\}, \{a, d, e\}$ $\{b, c, e\}$, $\{b, d, e\}$ does not form a triangle.
Suppose that neither of above cases takes place, that is, $b + c > d$, $c + d > e$ and at least one of inequalities $b + d > e$ and $a + b > d$ holds. We shall show that this is impossible.

iv. If $b + c > d$, $b + d > e$ then

$$a2 + b2 + c2 + d2 + e2 < ab + bc + ce + (b + c)d + (b + d)e.$$

Contradiction.

v. If $c + d > e$, $a + b > d$ then

$$a^2 + b^2 + c^2 + d^2 + e^2 < ab + bc + cd + (a + d)d + (c + d)e.$$

Contradiction.