MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 17, September 8, 2015[1]

1. If the number is made from $a \neq 1$, then $aaa\dots$ is divisible by a and thus not prime.

   Suppose the number has a non-prime number of digits, then we can factor the number of digit this number has into $q \times p$ where q and p are integers. Hence, we we can "split" the number up into q blocks of p-length digits; i.e

$$\underbrace{111\dots1}_{q\times q \text{ lots of 1's}} = \underbrace{\underbrace{111\dots1}_{p \text{ lots of 1's}} \times \underbrace{111\dots1}_{p \text{ lots of 1's}} \times\dots\times \underbrace{111\dots1}_{p \text{ lots of 1's}}}_{q \text{ lots of blocks}}.$$

   The RHS of the number above is divisible by $\underbrace{111\dots1}_{p \text{ lots of 1's}}$.

2. For a number to be divisible by 9, the sum of its digits must also be divisible by 9. If each digit of the number is even then so is the sum of its digits. So we start with smallest sum of digits that is divisible by 9 and even; this number is 18. It is easy to check that the number has at last 3-digits, so 288 is the smallest possible solution.

3. Since the RHS of $(m-8)(m-10) = 2^n$ is a power of 2, both $m-8$ and $m-10$ must be powers of 2. Also, the difference between $m-8$ and $m-10$ is 2, so the two solutions are $m-8 = 4$ and $m-8 = -2$.

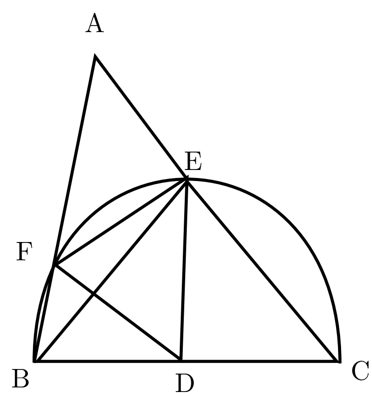
4. (a) Let try to see if there is a pattern. If a is number of nuts the children started with, and no nuts will be eaten. Then in four time steps, the number of nuts at each time step the children each will have is $a, 0, 0$, then $0, a/2, a/2$, then $a/4, 0, 3a/4$, and then $5a/8, 3a/4, 0$. It looks like at some times t in the future, one child will always have zero nuts, while the other two will have $xa/2^t$ and $ya/2^t$ nuts, where x and y are odd numbers. This can be proved using the standard induction argument.

   So at some time t, the number of nuts one of the child will have is $xa/2^t$ for some t, and we know this number must be an integer (the children are not allow to cut the nuts). Therefore, since x is odd, the number a must be divisible by 2^t for every t which is impossible.

[1]Some problems from UNSW's publication *Parabola* and the *Tournament of Towns in Toronto*.

(b) We show that we can not eat all nuts if at any moment the total number of nuts the children have is 3, then we are done because only one nut will be consumed at any time and we started with more than 3. Due to the argument in part (a), one child will have zero nuts, so the number of nuts each child have is $0, 2, 1$. Also, observe that the child with the highest number of nuts will be the next to do the dividing, so after another iteration, the number of nuts each child will have is $1, 0, 2$; this forms an endless loop.

5. $\triangle ABC$ can be equilateral, but we can construct an example where it is not: Let $\triangle DEF$ be an equilateral triangle. Construct a semicircle with centre D and radius DE. The diameter BC of the semicircle is perpendicular to DE, with F closer to B than to C. Since $DF = DE$, F also lies on this semicircle. Extend BF and CE to meet at A. Since $\angle BEC = 90° = \angle BFC$, BE and CF are indeed altitudes of triangle ABC. Since A lies on the extension of CE and DE is the perpendicular bisector of BC, $AB < AC$. Hence ABC is not equilateral.



6. If we compare the sum $1 + 2 + 3 + \ldots + 14 = 7 \times 15$ to $16 + 17 + 18 + \ldots 29 = 7 \times 45$, we see that latter is three times greater. Thus, all numbers < 15 must be place below the main diagonal, and all numbers > 15 above it. Hence, the remaining 29 lots of 15's must be place at the diagonal, which implies the central block is 15.

Senior Questions

1.
$$T_n\left(\frac{1}{T_1} + \frac{1}{T_2} + \ldots + \frac{1}{T_n}\right) = n^2$$
or
$$\frac{1}{T_1} + \frac{1}{T_2} + \ldots + \frac{1}{T_n} = \frac{2n}{2n+1}.$$
This can be proven using induction. The inductive step depends on
$$\frac{2n}{n+1} + \frac{a}{(n+1)(n+2)/2} = \frac{2(n+1)}{(n+1)+1}.$$

2. Let $(0,t)$ be the point of intersection of AB and CD. Then the equation of the line AB is given by $\frac{y}{x-t} = \frac{a^2-b^2}{a-b} = a+b$. That A lies on this line means that $\frac{a^2}{a-t} = a+b$. We have $a^2 = a^2ab - t(a+b)$ so that $t = \frac{ab}{a+b}$. Similarly, $t = \frac{cd}{c+d}$. Eliminating t, we have $abc + abd = d(ac+bc)$ so that $d = \frac{abc}{ac+bc-ab}$.

3. The graph of $f(x)$ is continuous, and have a number of turning points. Let $(x_i, f(x_i)$, $1 \le i \le n$ be the coordinate of these turning points. By symmetry, we can without loss of generality assume that the leading coefficient of $f(x)$ is positive. Suppose the degree of $f(x)$ is odd. Then $f(x)$ tends to ∞ as x tends to ∞, and $f(x)$ tends to $-\infty$ as x tends to $-\infty$. Let a be a real number such that $a > f(x_i)$ for all i. Then the equation $f(x) = a$ is satisfied by exactly one real number x. Suppose the degree of $f(x)$ is even. Then $f(x)$ tends to ∞ as x tends to $\pm\infty$. It follows that n is odd, so that there is a real number a for which $a = f(x_i)$ for an odd number of i. Then the equation $f(x) = a$ is satisfied by an even number of real numbers x where $x \ne x_i$, for $1 \le i \le n$, as well as by an odd number of x_i. Hence the equation is satisfied by an odd number of real numbers.