MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 7, June 9, 2015[1]

1. If we fix $x = 0$, then there are 100 choices for y. If we fix $x = 1$, then there are 99 choices for y, and so on. So the total number of ways to pick x and y such that $x + y \leq 100$ is equal to $1 + 2 + 3 + \ldots + 100 = \frac{100}{2}[2 + (100 - 1) \times 1] = 5050$.

2. It doesn't matter which prime you pick. If $p^2 + a^2 = b^2$ then
$$\begin{aligned} p^2 &= b^2 - a^2 \\ &= (b - a)(b + a). \end{aligned}$$
Because p is prime, the only divisor of p^2 is $1, p$ and p^2. Since a and b are integers, by the above equation, $b - a = 1$ and $b + a = p^2$, so that $\frac{a+b}{p} = p$.

3. The diagonal of the square is the diameter of the circle, hence the area of the circle is π.

   By Pythagoras the length of the sides of the square is $\sqrt{2}$. The area of the square is therefore 2.

   The sides of the square are the diameter of the smaller circles. The area of the four small half circles are therefore π.

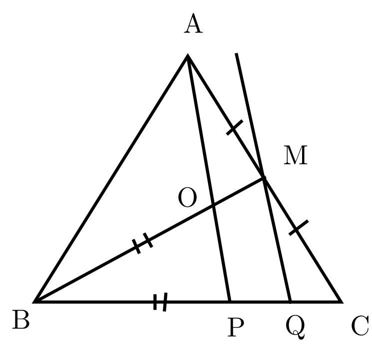
   Hence, the area of the shaded region is $\pi - (2 - \pi) = 2$.

4. Label the 21 people at the party by $a_1, a_2, \ldots, a_{21}$. Now a_1 knows at most four other people at the party, by renumbering we can assume that a_1 does not know $a_6, a_7, \ldots a_{21}$. By renumbering again, we can assume that a_6 knows at most four of $a_2, a_3, a_4, a_5, a_7, a_8, a_9, a_{10}$, therefore a_1 and a_6 does not know $a_{11}, a_{12}, \ldots, a_{21}$. Similarly by renumbering, a_1, a_6 and a_{11} does not know $a_{16}, a_{17} \ldots, a_{21}$, and a_1, a_6, a_{11} and a_{16} does not know a_{21}. It follows that a_1, a_6, a_{11}, a_{16} and a_{21} does not know each other mutually.

5. Set $g(x) = f(x) - 2015$, then a_1, a_2, a_3, a_4, a_5 are the roots of $g(x)$, therefore we can write $g(x) = c(x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)h(x)$, where c is some constant and $h(x)$ a polynomial.

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*.

Now the integral solutions to $f(x) = 2016$ are the integral solutions to $g(x) = 1$, but there is no integral solution to $g(x) = 1$, because in the expression $g(x) = c(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5)h(x)$, each $(x - a_i)$, $i = 1, 2, 3, 4, 5$ are distinct integers for any integer x. Also, $h(x)$ and c are integers for any integer x otherwise $f(x)$ will have non-integer coefficients; multiplying 7 integers in which at least 5 of are distinct can not give 1.



6. Draw a line parallel to AP that intersects the line BC at the point Q; see above. Note that the triangles $\triangle AOP$ and $\triangle AMQ$ are similar, so by triangles and ratios we have $|OM| = |PQ|$. Now to find $\frac{|OM|}{|PC|}$, all we have to do is work out what portion $|PQ|$ occupies $|PC|$.

   The triangles $\triangle ACP$ and $\triangle MCQ$ are similar, so by triangle and ratios we have $\frac{|AC|}{|PC|} = \frac{|MC|}{|QC|}$. But M is the midpoint of AC, which implies $|MC| = \frac{1}{2}|AC|$, so that

$$\frac{|AC|}{|PC|} = \frac{|MC|}{|QC|} = \frac{1}{2}\frac{|AC|}{|QC|}.$$

   It follows that $2|QC| = |PC|$, which implies $2|PQ| = |PC|$, and therefore $\frac{|OM|}{|PC|} = \frac{1}{2}$.

Senior Questions

1. I am not sure if there are suppose to be additional conditions on the roots or coefficients of $P(x)$, here is my reasoning to why I can not find such an N without additional assumptions: First we evaluate the polynomial at $x = 1$, this gives $P(1) = a_{99} + a_{98} + \ldots + a_2 + a_1 + 1 = 1 + \sum_{i=1}^{99} a_i$. Therefore, the problem is to find the largest integer N such that

$$\sum_{i=1}^{99} a_i = p(1) - 1 \geq 2(2^N - 1).$$

   So we look for the maximum lower bound for $P(1)$. Because the polynomial $P(x)$ has 100 roots, we can express it as $P(x) = (x + r_1)(x + r_2)\ldots(x + r_{99})(x + r_{100})$, where $r_1, r_2, \ldots, r_{99}, r_{100}$ are the roots of the $P(x)$ times -1. Now if we were to expand the RHS of $P(x) = (x+r_1)(x+r_2)\ldots(x+r_{99})(x+r_{100})$, then we can equate the coefficients of $P(x)$ by

$$\begin{aligned} a_{99} &= \sum_{i}^{100} r_i \\ a_{98} &= \sum_{i<j} r_i r_j \\ a_{97} &= \sum_{i<j<k} r_i r_j r_k \\ \vdots & \qquad \vdots \\ 1 &= r_1 r_2 \dots r_{99} r_{100}, \end{aligned}$$

where the notation $\sum_{i<j}$ means the product of all r_i with r_j over all index such that $i < j$, and similarly for $\sum_{i<j<k}$; that is the coefficient a_{99} of $P(x)$ is sum of the negative of roots of $P(x)$, the coefficient a_{98} is sum of product of two terms and so on. These forms the conditions on r_i.

Now we may set $r_1, r_2, \dots, r_{50} = y$ and $r_{51}, r_{52}, \dots, r_{100} = 1/y$, for some positive real number y, because $r_1 r_2 \dots r_{100} = 1$ and each coefficient $a_1, a_2, \dots a_{99}$ is positive. But then $P(1) = (1+y)^{50}(1+1/y)^{50} \geq (1+y)^{50}$; because y is arbitrary, I can not find such an N.

2. Let d be the greatest common divisor between x and y, write it as $\gcd(x, y) = d$. Then we have $x = d \times x'$ and $y = d \times y'$, where x' and y' are some integers such that $\gcd(x', y') = 1$. Now in order to show that $x + y$ is a square, we just need to show that $x' + y' = d$, because this implies $x + y = d^2$.

   We can rewrite $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ as $z(x + y) = xy$ or equivalently $z(x' + y') = dx'y'$. Since $\gcd(x, y, z) = 1$, $\gcd(d, z) = 1$. Furthermore, x' does not divide y' and visa versa, therefore $\gcd(x' + y', x') = \gcd(x' + y', y') = 1$. It follows from the equation $z(x' + y') = dx'y'$ that x' and y' must divide z, so we have $x'y' = z$, which implies $x' + y' = d$.

3. We start by trying a few values of n to see if we can spot a pattern.

$$\begin{aligned} n &= 1, & 14^1 + 11 = 25 = 5(5) \\ n &= 2, & 14^2 + 11 = 207 = 3(69) \\ n &= 3, & 14^3 + 11 = 25 = 5(551) \\ n &= 2, & 14^4 + 11 = 207 = 3(12809) \end{aligned}$$

   It seems like when n is odd, $14^n + 11$ is divisible by 5, and when n is even , $14^n + 11$ is divisible by 3.

   If n is even then $14^n = 14^{2k} = 196^k$. As 196 has remainder 1 when divided by 3, it follows that 196^k has remainder 1 when divided 3. Therefore $142^k + 11$ is divisible by 3.

   If n is odd, then $14^n = 14^{2k+1} = 14 \times 14^{2k} = 14 \times 196^k$. As 196 as remainder 1 when divided by 5, it follows that 196^k also has remainder 1 when divided by 5, and 14×196^k has remainder 4 when divided by 5. Therefore $14^{2k} + 1 + 11$ is divisible by 5.

Hence $14n + 11$ is divisible by 5 and 3 alternately, and can never be prime.