MATHEMATICS ENRICHMENT CLUB.
Solutions Sheet 8, June 16, 2015[1]

1. We have

$$x^2 - y^2 = 1999$$
$$(x-y)(x+y) = 1999,$$

and since 1999 is prime, either $(x-y) = \pm 1$, $(x+y) = \pm 1999$ or $(x+y) = \pm 1$, $(x-y) = \pm 1999$; there is a total of four integral solutions to both cases.

2. There are two solutions to this problem: One uses the perimeter of $\triangle ABC$, the other uses the area of $\triangle ABC$.

   (a) Hint: Let the point of intersection between the circle and the side AB be P. Then the radius of the inscribed circle is $|AP|$, and the line PB is tangent to the circle at the point P.

   (b) Hint: Let M be the middle of the inscribed circle. Then the triangles $\triangle ABM$, $\triangle BCM$ and $\triangle CAM$ all have height equal to the radius of the inscribed circle.

3. A neat trick is to express N as

$$\underbrace{333\ldots333}_{61\times 3's} = \frac{3}{9}\left(\underbrace{999\ldots999}_{61\times 9's}\right) = \frac{3}{9}(10^{61}-1).$$

Similarly, $M = \underbrace{666\ldots666}_{62\times 6's} = \frac{6}{9}(10^{62}-1)$. Now

$$\begin{aligned} N \times M &= \frac{2}{9}(10^{61}-1)(10^{62}-1) \\ &= \frac{2}{9}(10^{61}-1)\times 10^{62} - \frac{2}{9}(10^{61}-1) \\ &= \underbrace{222\ldots222}_{61\times 2's}\underbrace{000\ldots000}_{62\times 0's} - \underbrace{222\ldots222}_{61\times 2's} \\ &= \underbrace{222\ldots222}_{60\times 2's}19\underbrace{777\ldots777}_{60\times 7's}8. \end{aligned}$$

[1] Some problems from UNSW's publication *Parabola*.

It is easy to to compute the sum of digits on the last line of the above equation; it is 558.

4. First we note that for positive integers m, n, k and r, if $a = mk + r$, then $a^x = nk + r^x$ (you may want to show this is true). Now, since $a = a-1+1$, we have $a^x = (a-1)m_1+1^x$ for some positive integer m_1, thus $r_1 = 1^x = 1$. Similarly, since $a = a + 1 - 1$ and x is odd, we have $a^x = (a + 1)m_2 - 1^x = (a + 1)m_2 - 1 = (a + 1)(m_2 - 1) + a$ for some integer m_2, thus $r_2 = a$. Hence, we can conclude that $r_1 + r_2 = a + 1$.

5. We can write $x = n+d$, where n is the integral part of x and d the decimal part. Then $[2x] + [4x] + [6x] + [8x] = 20n + [2d] + [4d] + [6d] + [8d]$. We scan over the range of d; that is $0 < d < 1$ to see what positive integer under 1001 can be expressed in the form of $[2x] + [4x] + [6x] + [8x]$. For example

$[2x]$	$+$	$[4x]$	$+$	$[6x]$	$+$	$[8x]$		
0	$+$	0	$+$	0	$+$	1	$= 1,$	if $\frac{1}{8} \le d < \frac{1}{6}$.
0	$+$	0	$+$	1	$+$	1	$= 2,$	if $\frac{1}{6} \le d < \frac{1}{4}$.
0	$+$	1	$+$	1	$+$	2	$= 4,$	if $\frac{1}{4} \le d < \frac{1}{3}$.
0	$+$	1	$+$	2	$+$	2	$= 5,$	if $\frac{1}{3} \le d < \frac{3}{8}$.
0	$+$	1	$+$	2	$+$	3	$= 6,$	if $\frac{3}{8} \le d < \frac{1}{2}$.

If we continue with the above calculations, the results are the numbers ending in $3, 7, 8$ or 9 can not be expressed in the form $[2x] + [4x] + [6x] + [8x]$. There are 100 numbers ending in 3 that is less than 1001, and similarly for $7, 8$ and 9. Hence there are 600 numbers that can be put into the required form.

6. Let d be the number of kilometres traveled before the tyre switch is made. Then $\frac{d}{x}$ is the proportion of wear on the font tyre before the switch, hence they will travel a further $\left(1 - \frac{d}{x}\right) y$ kilometres before the tyres are retired. So the total distance traveled by the font tyre is $d + \left(1 - \frac{d}{x}\right) y$. Similarly, the total distance traveled by the rear tyre is $d + \left(1 - \frac{d}{y}\right) x$.

Suppose the claim of the advertisement is true, then we must have the following system of inequalities

$$\begin{aligned} d + \left(1 - \tfrac{d}{x}\right) y &\ge \tfrac{x+y}{2} \\ d + \left(1 - \tfrac{d}{y}\right) x &\ge \tfrac{x+y}{2}. \end{aligned} \qquad (*)$$

Rearranging (*) gives

$$\begin{aligned} d\left(1 - \tfrac{y}{x}\right) &\ge \tfrac{x-y}{2} \\ d\left(1 - \tfrac{x}{y}\right) &\ge \tfrac{y-x}{2}, \end{aligned}$$

then by using the assumption that $x < y$, we have

$$d \leq \tfrac{x-y}{2} \times \left(1 - \tfrac{y}{x}\right)^{-1} = \tfrac{x}{2}$$
$$d \geq \tfrac{x-y}{2} \times \left(1 - \tfrac{x}{y}\right)^{-1} = \tfrac{y}{2}.$$

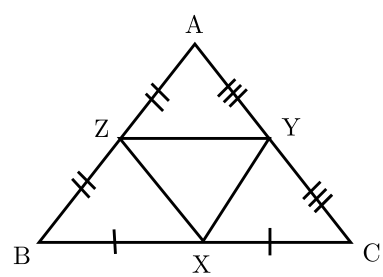
The last system of inequality does not hold because $x < y$, so we have a contradiction to the advertisement's claim.

Senior Questions

1. Since $\alpha > 0$, $\left(\alpha + \frac{1}{\alpha}\right)^2 = \alpha^2 + \frac{1}{\alpha^2} + 2 \geq 2$. Similarly, $\left(\beta + \frac{1}{\beta}\right)^2 \geq 2$. Therefore, if r_1 and r_2 are the roots of f (assuming $r_1 \geq r_2$ wlog). Then $r_1 \geq 2$ and $r_2 < 0$, so that $r_1 r_2 = c - 3 < 0$, which implies $c < 3$.

   To get the lower bound on c, we use the quadratic formula $2 \leq r_1 = (c+1) + \sqrt{(c+1)^2 - 4(c-3)}$. Solving gives $-2 \leq c$.

2. Lets start by looking at the extreme case $BX = XC, CY = YA, AZ = ZB$; as shown below. By the Midpoint Theorem, the line BC is parallel to ZY, and the line AC is parallel to ZX. Therefore $ZYBX$ forms a parallelogram, which implies $|BZ| = |YX|$ and $|ZY| = |BX|$. By similar arguments, $ZYXC$ forms a parallelogram, which implies $|ZX| = |YC|$. Hence, the triangles $\triangle ZYZ$, $\triangle BXZ$, $\triangle XCY$ and $\triangle XYZ$ are identical, so that the area of $\triangle XYZ$ is exactly one quarter.



   (a) If we fix Z and Y (recall that $|AZ| \leq |ZB|$ and $|CY| \leq |AY|$) , then as we move the point X towards the midpoint of BC, the area of the triangle $\triangle XYZ$ is either decreasing or constant. More generally, if we fix any two points of Z, Y or X and move the other towards the midpoint of the side they are located on, then the area of the triangle $\triangle XYZ$ is decreasing or constant. The smallest possible area of $\triangle XYZ$ occurs when Z, Y and X are the midpoints of the sides of the triangle $\triangle ABC$; in this case the area of $\triangle XYZ$ is exactly a quarter of $\triangle ABC$.

   (b) This follows immediately from the results of part (a).

3. Square both sides of the equation $\sqrt{a} - b = \sqrt{c}$ and rearranging gives

$$\sqrt{c} = \frac{a - b^2 - c}{2b}.$$

Since the RHS of the above equation is rational, $\sqrt{c}$ must be rational. Write $\sqrt{c} = x/y$, where x and y are integers with greatest common multiplier one. Then $c = x^2/y^2$, and greatest common multiplier between x^2 and y^2 is one. Since c is an integer, x^2 must be divisible by y^2, which can only happen if $y^2 = 1$, because the greatest common multiplier between x^2 and y^2 is one. Hence $c = x^2$, so that c is a perfect square.

If c is a perfect square, then the equation $\sqrt{a} - b = \sqrt{c}$ implies that a is also a perfect square.

4.