MATHEMATICS ENRICHMENT CLUB.  
Solutions Sheet 8, June 16, 2015

1. We have

\[ x^2 - y^2 = 1999 \]
\[ (x - y)(x + y) = 1999, \]

and since 1999 is prime, either \((x - y) = \pm 1, (x + y) = \pm 1999\) or \((x + y) = \pm 1, (x - y) = \pm 1999\); there is a total of four integral solutions to both cases.

2. There are two solutions to this problem: One uses the perimeter of \(\triangle ABC\), the other uses the area of \(\triangle ABC\).

(a) Hint: Let the point of intersection between the circle and the side \(AB\) be \(P\). Then the radius of the inscribed circle is \(|AP|\), and the line \(PB\) is tangent to the circle at the point \(P\).

(b) Hint: Let \(M\) be the middle of the inscribed circle. Then the triangles \(\triangle ABM\), \(\triangle BCM\) and \(\triangle CAM\) all have height equal to the radius of the inscribed circle.

3. A neat trick is to express \(N\) as

\[
\frac{333 \ldots 333}{61 \times 3's} = \frac{3}{9} \left( \frac{999 \ldots 999}{61 \times 9's} \right) = \frac{3}{9}(10^{61} - 1). 
\]

Similarly, \(M = \frac{666 \ldots 666}{62 \times 6's} = \frac{6}{9}(10^{62} - 1)\). Now

\[
N \times M = \frac{2}{9}(10^{61} - 1)(10^{62} - 1) = \frac{2}{9}(10^{61} - 1) \times 10^{62} - \frac{2}{9}(10^{61} - 1)
\]
\[
= \frac{222 \ldots 222 000 \ldots 000}{61 \times 2's} - \frac{222 \ldots 222}{61 \times 2's}
\]
\[
= \frac{222 \ldots 222 19 777 \ldots 777}{60 \times 2's} 8.
\]

\[ ^1 \text{Some problems from UNSW’s publication } \textit{Parabola}. \]
It is easy to compute the sum of digits on the last line of the above equation; it is 558.

4. First we note that for positive integers $m, n, k$ and $r$, if $a = mk + r$, then $a^x = nk + r^x$ (you may want to show this is true). Now, since $a = a-1+1$, we have $a^x = (a-1)m_1 + 1^x$ for some positive integer $m_1$, thus $r_1 = 1^x = 1$. Similarly, since $a = a + 1 - 1$ and $x$ is odd, we have $a^x = (a + 1)m_2 - 1^x = (a + 1)m_2 - 1 = (a + 1)(m_2 - 1) + a$ for some integer $m_2$, thus $r_2 = a$. Hence, we can conclude that $r_1 + r_2 = a + 1$.

5. We can write $x = n + d$, where $n$ is the integral part of $x$ and $d$ the decimal part. Then $[2x] + [4x] + [6x] + [8x] = 20n + [2d] + [4d] + [6d] + [8d]$. We scan over the range of $d$; that is $0 < d < 1$ to see what positive integer under 1001 can be expressed in the form of $[2x] + [4x] + [6x] + [8x]$. For example

<table>
<thead>
<tr>
<th>$2x$</th>
<th>$4x$</th>
<th>$6x$</th>
<th>$8x$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>0</td>
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<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

If we continue with the above calculations, the results are the numbers ending in 3, 7, 8 or 9 can not be expressed in the form $[2x] + [4x] + [6x] + [8x]$. There are 100 numbers ending in 3 that is less than 1001, and similarly for 7, 8 and 9. Hence there are 600 numbers that can be put into the required form.

6. Let $d$ be the number of kilometres traveled before the tyre switch is made. Then $\frac{d}{x}$ is the proportion of wear on the font tyre before the switch, hence they will travel a further $(1 - \frac{d}{x}) y$ kilometres before the tyres are retired. So the total distance traveled by the font tyre is $d + (1 - \frac{d}{x}) y$. Similarly, the total distance traveled by the rear tyre is $d + \left(1 - \frac{d}{y}\right)x$.

Suppose the claim of the advertisement is true, then we must have the following system of inequalities

$$d + \left(1 - \frac{d}{x}\right)y \geq \frac{x+y}{2}$$
$$d + \left(1 - \frac{d}{y}\right)x \geq \frac{x+y}{2}.$$

Rearranging (*) gives

$$d\left(1 - \frac{y}{x}\right) \geq \frac{x-y}{2},$$
$$d\left(1 - \frac{x}{y}\right) \geq \frac{y-x}{2}.$$
then by using the assumption that \( x < y \), we have

\[
\begin{align*}
d &\leq \frac{x - y}{2} \times \left(1 - \frac{y}{x}\right)^{-1} = \frac{x}{2} \\
d &\geq \frac{x - y}{2} \times \left(1 - \frac{x}{y}\right)^{-1} = \frac{y}{2}.
\end{align*}
\]

The last system of inequality does not hold because \( x < y \), so we have a contradiction to the advertisement’s claim.

**Senior Questions**

1. Since \( \alpha > 0 \), \( (\alpha + \frac{1}{\alpha})^2 = \alpha^2 + \frac{1}{\alpha^2} + 2 \geq 2 \). Similarly, \( (\beta + \frac{1}{\beta})^2 \geq 2 \). Therefore, if \( r_1 \) and \( r_2 \) are the roots of \( f \) (assuming \( r_1 \geq r_2 \) wlog). Then \( r_1 \geq 2 \) and \( r_2 < 0 \), so that \( r_1 r_2 = c - 3 < 0 \), which implies \( c < 3 \).

   To get the lower bound on \( c \), we use the quadratic formula \( 2 \leq r_1 = (c + 1) + \sqrt{(c + 1)^2 - 4(c - 3)} \). Solving gives \( -2 \leq c \).

2. Let’s start by looking at the extreme case \( BX = XC, CY = YA, AZ = ZB \); as shown below. By the Midpoint Theorem, the line \( BC \) is parallel to \( ZY \), and the line \( AC \) is parallel to \( ZX \). Therefore \( ZYBX \) forms a parallelogram, which implies \( |BZ| = |YX| \) and \( |ZY| = |BX| \). By similar arguments, \( ZYXC \) forms a parallelogram, which implies \( |ZX| = |YC| \). Hence, the triangles \( \triangle ZYZ \), \( \triangle BXZ \), \( \triangle XCY \) and \( \triangle XYZ \) are identical, so that the area of \( \triangle XYZ \) is exactly one quarter.

![Diagram](https://via.placeholder.com/150)

(a) If we fix \( Z \) and \( Y \) (recall that \( |AZ| \leq |ZB| \) and \( |CY| \leq |AY| \)), then as we move the point \( X \) towards the midpoint of \( BC \), the area of the triangle \( \triangle XYZ \) is either decreasing or constant. More generally, if we fix any two points of \( Z, Y \) or \( X \) and move the other towards the midpoint of the side they are located on, then the area of the triangle \( \triangle XYZ \) is decreasing or constant. The smallest possible area of \( \triangle XYZ \) occurs when \( Z, Y \) and \( X \) are the midpoints of the sides of the triangle \( \triangle ABC \); in this case the area of \( \triangle XYZ \) is exactly a quarter of \( \triangle ABC \).

(b) This follows immediately from the results of part (a).

3. Square both sides of the equation \( \sqrt{a} - b = \sqrt{c} \) and rearranging gives

\[
\sqrt{c} = \frac{a - b^2 - c}{2b}.
\]
Since the RHS of the above equation is rational, $\sqrt{c}$ must be rational. Write $\sqrt{c} = x/y$, where $x$ and $y$ are integers with greatest common multiplier one. Then $c = x^2/y^2$, and greatest common multiplier between $x^2$ and $y^2$ is one. Since $c$ is an integer, $x^2$ must be divisible by $y^2$, which can only happen if $y^2 = 1$, because the greatest common multiplier between $x^2$ and $y^2$ is one. Hence $c = x^2$, so that $c$ is a perfect square.

If $c$ is a perfect square, then the equation $\sqrt{a} - b = \sqrt{c}$ implies that $a$ is also a perfect square.