MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 1, May 5, 2015

1. Let $x = 0.284284284\ldots$, then
$$\begin{aligned} 1000x &= 284.284284284\ldots \\ &= 284 + x, \end{aligned}$$
thus $x = 284/999$.

2. We can write the finite sum as
$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{10100} = 1 + \sum_{n=2}^{101} \frac{1}{n \times (n-1)}$$
Using the given formula,
$$\begin{aligned} 1 + \sum_{n=2}^{101} \frac{1}{n \times (n-1)} &= 1 + \sum_{n=2}^{101} \frac{n-1}{n} - \frac{n-2}{n-1} \\ &= 1 + \sum_{n=2}^{101} \frac{n-1}{n} - \sum_{n=1}^{100} \frac{n-1}{n} \\ &= 1 + \frac{100}{101}. \end{aligned}$$

3. Let S be the number of members that plays Soccer.

   (a) If we add the number of members that plays either Basketball, Cricket or Soccer, we would end up with a number that is greater than the total number of members in the sports club, because we have double counted the number of members that plays two sports *only*, and triple counted the number of members that plays *all* three.

   So to balance this out we need to subtract the double/triple counts: We know that 10 plays *all* three sports, so these member we triple counted. There is 60 members that plays two or more sports, and 10 that plays all three, therefore there is $60 - 10 = 50$ members that plays two sports *only*.

   The balanced equation is then
$$163 = S + 100 + 73 - 50 - 2(10),$$
   which gives $S = 60$.

(b) The number of members that plays both Basketball and Cricket but not Soccer is $25 - 10 = 15$, therefore $60 - 15 = 45$ members plays Soccer and Basketball or Soccer and Cricket or all three sports. Since $S = 60$, $60 - 45 = 15$ of these members plays Soccer only.

4. (a) Here $|QC|$ means the length of QC. By construction, the length of AP is b; that is$|AP| = b$. Since the point Q is the intersection of the tangent PQ and CQ of the same circle arc PC, $|PQ| = |CQ|$ (you may want to prove this as an exercise). So the problem is reduced to finding $|PQ|$. Note that BPQ is an isosceles triangle, so $|CQ| = |PQ| = |PB| = a - b$.

   (b) Suppose we can find whole numbers a and b such that the ratio a/b is in its simplest form and $a/b = \sqrt{2}$. We'll use the result of part a) to produce a contradiction on the condition that the ratio a/b is in its simplest forms.

   As noted in part a), the triangle BPQ is an isosceles triangle. Moreover, BPQ is similar to the triangle ABC; so that the ratio

$$\frac{|AB|}{|AC|} = \frac{|BQ|}{|BP|} = \frac{|BC| - |QC|}{|BA| - |PA|},$$

   holds. Since ABC is a right angled isosceles triangle, we have$|AB|/|AC| = \sqrt{2} = a/b$. So the above ratio equation can be expressed in terms of a and b as

$$\frac{a}{b} = \frac{b - (a - b)}{a - b},$$

   so we have just found $\sqrt{2} = (2b - a)/(a - b)$. Now we can argue that if a and b are whole numbers, then $2b - a$ is a whole number smaller than a, and $a - b$ is a whole number smaller than b, this contradicts a/b being in the simplest form.

5. (a) Let $a_1, a_2, \ldots a_k$ be the digits of a k digit long whole number x. Then we can expression $x = a_k a_{k-1} \ldots a_2 a_1 = 10^k a_k + 10^{k-1} a_{k-1} + \ldots + 10^2 a_3 + a_2 a_1$. Since 10^i is divisible by 4 for $i = 2, 3, \ldots k$, for x to be divisible by 4, then so is $a_2 a_1$; the number formed by the last two digits of x.

   (b) Similar to a), first express the k digit long number as $x = a_k a_{k-1} \ldots a_2 a_1 = 10^k a_k + 10^{k-1} a_{k-1} + \ldots + 10^2 a_3 + 10 a_2 + a_1$. Let y be the number formed by the sum of all of the digits of x; that is $y = a_1 + a_2 + \ldots + a_{k-1} + a_k$. Consider the difference $x - y = (10^k - 1)a_k + (10^{k-1})a_{k-1} + \ldots + 99a_3 + 9a_2$, then $x - y$ is divisible by 9, so if x is divisible by 9, then so must y.

6. Let $x = n(n+1)(n+2)$, then x is the product of 3 consecrative numbers. Hence, x is divisible by 2 and 3, which means x^2 is a divisible by 4 and 9. We can now use the results of Q5 to find the missing digit $*$:

   Since x^2 is divisible by 9, the result of 5.a) says that the sum of the digits of x^2; that is $4 + 8 + 1 + 2 + 7 + 3 + 5 + 6 + 3 + * + 6 = 45 + *$ must be divisible by 9, so the digit $*$ must be either 0 or 9.

   Also, x^2 is divisible by 4, so by the result of 5.b) the number $*6$ formed by the last two digits of x^2 must be divisible by 4, this can only happen if $*6 = 40$, therefore $* = 0$.

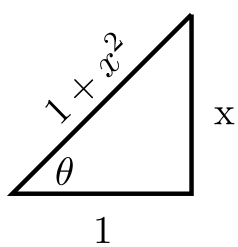
Senior Questions

1. Let $\theta = \tan^{-1}(x)$, then $x = \tan(\theta)$, and

$$\frac{dx}{d\theta} = \frac{1}{\cos^2\theta}.$$

Now by using the picture below $\cos^2\theta = 1/(1+x^2)$, therefore $dx/d\theta = 1 + x^2$, thus

$$\frac{d\theta}{dx} = \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$



2. By polynomial long division (see for example `http://en.wikipedia.org/wiki/Polynomial_long_division`), we can express

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots.$$

Since the LHS of the above equation is $\frac{d}{dx}\tan^{-1}(x)$ by Q1, integrating both sides gives

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

3. This solution provide by Adam Solomon:

$$\begin{aligned} n(n+1)(n+2)(n+3)+1 &= (n^2+3n)(n^2+3n+2)+1 \\ &= (n^2+3n)(n^2+3n+1)+n^2+3n+1 \\ &= (n^2+3n+1)^2. \end{aligned}$$

To calculate $\sqrt{(31)(30)(29)(28)+1}$, set $n = 28$ in n^2+3n+1.