



MATHEMATICS ENRICHMENT CLUB.

Problem Sheet 4, May 19, 2015¹

1. Let a and b be positive integers such that $2^a - 2^b = 2016$. Find the value of $a + b$.
2. Let $ABCD$ be a square, with M and N the mid points of the sides BC and AD respectively. K is an arbitrary point on the extension of the diagonal AC beyond A . The segment KM intersects the side AB at some point L . Prove that $\angle KNA = \angle LNA$.
3. Find the smallest positive integer n such that $\frac{1}{3}n$ is a perfect cube, $\frac{1}{5}n$ a perfect fifth power and $\frac{1}{7}n$ a perfect seventh power.

4. Simplify

$$\left(\frac{2^3 - 1}{2^3 + 1}\right) \left(\frac{3^3 - 1}{3^3 + 1}\right) \left(\frac{4^3 - 1}{4^3 + 1}\right) \cdots \left(\frac{n^3 - 1}{n^3 + 1}\right).$$

5. Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b\sqrt{c}$, where a, b , and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$. Hint: Interpret this problem geometrically.
6. In how many ways can we choose n integers x_1, x_2, \dots, x_n such that each is 0, 1 or 2 and their sum is even?

¹Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*.

Senior Questions

1. Given that a , b , and c are positive integers, solve
 - (a) $a!b! = a! + b!$
 - (b) $a!b! = a! + b! + 2^c$
 - (c) $a!b! = a! + b! + c!$
2.
 - (a) Prove that for $n \geq 3$, $(n + 1)! > (n - 2)(1! + 2! + \dots + n!)$.
 - (b) Use part (a) or otherwise, show that for $n \geq 3$, $(n + 1)!$ is not divisible by $1! + 2! + \dots + n!$.