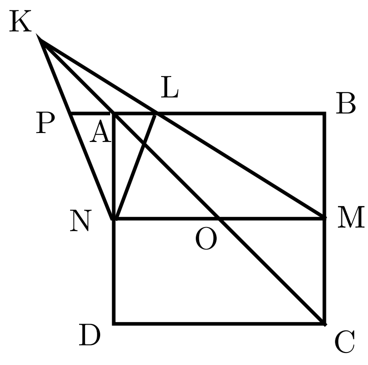
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 4, May 19, 2015[1]

1. First write $2016 = 2^5 3^2 7$, then divide both sides by 2^b we get

$$\begin{aligned} 2^{a-b} - 1 &= 2^{5-b}3^2 7 \\ 2^{a-b} &= 2^{5-b}3^2 7 + 1. \end{aligned} \tag{1}$$

Since $2^a - 2^b = 2016 > 0$, $a > b$, which implies the LHS of equation (1) is an even number. For the RHS of (1) to be even, we must have $b = 5$. Substituting $b = 5$ into (1), then $2^{a-5} = 64$, solving to obtain $a = 11$.

2. Let O be the midpoint of NM, extend the line AB so that it intercepts KN at the point P; see below. Since NM and PL are parallel and O is the mid point of NM, A is the midpoint of PL (this is a special case of the intercept theorem `http://en.wikipedia.org/wiki/Intercept_theorem`). Therefore the triangles PNA and ANL are congruent to each other, hence $\angle PNA = \angle ANL$.



3. We can write n as $n = 3^a 5^b 7^c \times N$, where the number N has no factors of 3, 5 or 7. Then $\frac{1}{3}n = 3^{a-1}5^b 7^c \times N$, $\frac{1}{5}n = 3^a 5^{b-1} 7^c \times N$ and $\frac{1}{7}n = 3^a 5^b 7^{c-1} \times N$. Because we are looking minimal N, we may as well set $N = 1$. So for $\frac{1}{3}n$ to be a perfect cube, $\frac{1}{5}n$ to be a perfect fifth power and $\frac{1}{7}$ to be a perfect seventh power, we must have $a - 1$ a multiple of 3 and a a multiplied of $5, 7$; the smallest such a is 70. To find n, repeat this argument to obtain b and c.

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

4. We have

$$k^3 - 1 = (k-1)(k^2+k+1) = (k-1)(k(k+1)+1)$$

and

$$k^3 + 1 = (k+1)(k^2-k+1) = (k+1)(k(k-1)+1).$$

Therefore the numerator of the given product contains the factors $1, 2, 3, \ldots, n-1$ and the denominator contains $3, 4, 5, \ldots, n+1$. Most of these cancel and we are left with $2/n(n+1)$. The numerator also contains factors $2 \times 3 + 1, 3 \times 4 + 1, \ldots, n(n+1)+1$,and the denominator $1 \times 2 + 1, 2 \times 3 + 1, \ldots, (n1)+1$; again most cancel and there remains $(n(n+1)+1)/(1 \times 2 + 1)$. Combining all these results gives

$$\frac{2^3-1}{2^3+1}\frac{3^3-1}{3^3+1}\frac{4^3-1}{4^3+1}\cdots\frac{n^3-1}{n^3+1} = \frac{2}{n(n+1)}\frac{n(n+1)+1}{1 \times 2+1} = \frac{2}{3}\frac{n^2+n+1}{n^2+n}.$$

5. Let M_1 and M_2 be the two mathematicians. We can plot the arrival time of M_1 and M_2 on the x-y plane, with x-axis representing the arrival time of M_1, and y-axis the arrival time of M_2; see figure 1. Each mathematician stays in the tea room for exactly m minutes, so we know that if M_1 arrives first (say at 9 a.m.) then M_2 will run into M_1 in the cafeteria if M_2's arrival time is within m minutes of M_1; this is represented by the $m \times m$ square box in the bottom left of the plot. Over the break of 60 minutes, we get a shaded region as shown in figure 1.

The probability that either mathematician arrives while the other is in the cafeteria is 40%, thus the non-shaded region is 60% of the total area of the big square. So we have

$$\frac{(60-m)^2}{60^2} = 0.6$$
$$m = 60 - 12\sqrt{15},$$

therefore, $a + b + c = 87$.

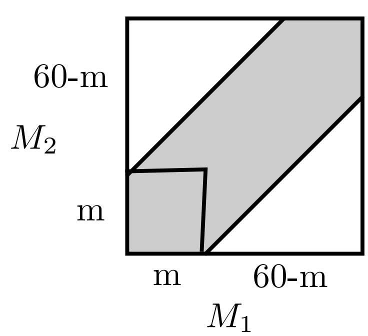


Figure 1: shaded area represents either mathematician arrives while the other is in the cafeteria

6. Let $f(n)$ be the number of ways we can choose these n integers. We can try to workout what $f(n+1)$ is; that is the number of ways to choose $x_1, x_2, \ldots, x_n, x_{n+1}$ such that each is $0, 1$ or 2 and their sum even.

Suppose we have n integers, $x_1, \ldots x_n$ from the list $0, 1, 2$ such that their sum is even. We know there is $f(n)$ ways to choose these n numbers, and we can either pick x_{n+1} to be 0 or 2 so that the sum of $x_1, \ldots, x_{n+1}$ is even; the total number of ways we can pick these $n+1$ integers is $2f(n)$.

On the other hand, if the initial n integers, $x_1, \ldots x_n$ from the list $0, 1, 2$ is odd, then there is $3^n - f(n)$ ways to choose these n numbers, and we can only pick $x_{n+1} = 1$ so that the sum of $x_1, \ldots x_{n+1}$ is even; the total number of ways we can pick these $n+1$ integers is $3^n - f(n)$

Combining both cases, we have the recursive relation $f(n+1) = 3^n + f(n)$. Since it is straightforward to workout $f(1) = 2$, we can find $f(n)$.

Senior Questions

1. Given that a, b, and c are positive integers, solve

   (a) If $a > b$, then dividing both sides by $a!$, we have

   $$b! = \frac{b!}{a!} + 1,$$

   the LHS of the above equation is an integer, while the RHS is not; we have a contradiction on the condition $a > b$. We can apply the same arguments to get $a \not< b$, so that $a = b$. The only solution is then $a = b = 2$.

   (b) Notice this equation is symmetric in a and b, so we can assume without loss of generality $a \geq b$. Dividing through by $b!$, then

   $$a! = \frac{a!}{b!} + 1 + \frac{2^c}{b!}. \tag{2}$$

   The LHS of equation (2) is an integer and $a!/b!$ is an integer, therefore $2^c/b!$ must be an integer, this implies b is either 1 or 2. Also, the RHS of (2) is the sum of 3 integers, so $a!$ must contain a factor of 3; $a \geq 3$.

   If $b = 1$ then $a! = a! + 1 + 2^c$, which implies $2^c + 1 = 0$; there is no solution for c, so $b \neq 1$. Therefore $b = 2$.

   If $a > 3$, then $a!/2$ is even, so $2^{c-1} = 1$. But then we get $a!/2 = 2$, which has no solution for a.

   Therefore, we conclude that $a = 3$ and $b = 2$, therefore $c = 2$.

   (c)

2. (a) The inequality holds for $n = 3$. Assume $n! > (n-2)(1! + 2! + \ldots (n-)1!)$ and note that $2(n-2) \geq n-1$ for $n \geq 3$, therefore

   $$\begin{aligned}(n+1)! &= (n-1)n! + 2n! \\ &> (n-1)n! + 2(n-2)(1! + 2! + \ldots (n-1)!) \\ &\geq (n-1)(1! + 2! + \ldots + n!),\end{aligned}$$

   so the inequality holds for all n by standard induction arguments.

(b) $(n+1)! < n(1! + 2! + \ldots + n!)$ because

$$\begin{aligned}(n+1)! &= (n+1)n! \\ &= nn! + n! \\ &= n(n! + (n-1)!) \\ &< n(1! + 2! + \ldots + n!).\end{aligned}$$

Therefore, combining with the result of (a),

$$n < \frac{(n+1)!}{1! + 2! + \ldots + n!} < n+1.$$

So $(n+1)!$ divided by $1! + 2! + \ldots n!$ is a number that is strictly between n and $n+1$; $1! + 2! + \ldots n!$ does not divide $(n+1)!$.