MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 16, September 5, 2016

1. Let n be a positive integer. If the polynomial
$$(x+1)(x+2)(x+3)\dots(x+n)$$
is expanded, find the sum of the coefficients of odd powers of x.

2. Four points are located in a plane. For each point, the sum of the distances to the other three is calculated; and these four sums are found to be the same. Determine all possible configurations of the four points.

3. Prove that if $n \geq 4$ then any triangle can be dissected into n isosceles triangles.

4. Each corner of a cube is labelled with a number. In each step, each number is replaced with the average of the labels of the three adjacent corners. All eight numbers are replaced simultaneously. After ten steps, all labels are the same as their respective initial values. Does it necessarily follow that all eight numbers are equal initially?

5. In a certain big city, all the streets go in one of two perpendicular directions. During a drive in the city, a car does not pass through any place twice, and returns to the parking place along a street from which it started. If it has made 100 left turns, how many right turns must it have made?

6. Steve's watch works properly, but has no markings on its face. The hour, minute and second hands have distinct lengths, and they move uniformly. Steve claims that since none of the mutual positions of the hands repeats twice in the period between $8:00$ and $19:59$, he can use his watch to tell the time during the day. Is his assertion true?

Senior Questions

1. Unit cubes are arranged into an $20 \times 18 \times 15$ block. A straight line is drawn from one corner of the block to the diagonally opposite corner. How many unit cubes does the line pass through?

2. A collection of 2016 numbers consists of one zero and 2015 ones.

   (a) It is permitted to choose any two numbers from the collection and replace each of them by the average of the two. Is it possible by repeating this operation to obtain a collection in which all 1995 numbers are the same?

   (b) It is permitted to choose any two or more of the numbers (but not the whole collection) and replace each of them by the average of the chosen numbers. Is it possible to make all the numbers equal?

3. The graphs of four functions of the form $y = x^2 + ax + b$, where a and b are real coefficients, are plotted on the coordinate plane. These graphs have exactly four points of intersection, and at each one of them, exactly two graphs intersect. Prove that the sum of the leargest and the smallest x-coordinate of the points of intersection is equal to the sum of the other two.