MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 17, September 12, 2016

1. Find all positive integer solution pairs (x, y) for the equation

$$x^2 + xy - y - 1 = 2016.$$

2. There are ten cards with the number a on each, ten with b and ten with c, where a, b and c are distinct real numbers. For every five cards, it is possible to add another five cards so that the sum of the numbers on these ten cards is 0. Prove that one of a, b and c is 0.

3. One day, Albert from Anastasia and Associates Attorneys and Betty from Bartholomew and Brothers Barristers leave at the same time to deliver messages to the other legal firm. They follow the same route, walking at constant but different speeds and pass each other when Albert has walked A metres. After delivering their messages, and each waiting 10 minutes for a reply, they return to their own firm at the same speeds as before and over the same route, this time passing when Betty is still B metres from home. How far did each walk? If $A < B$ who walks the faster?

4. In a sequence of distinct positive integers, each term except the first is either the arithmetic mean or the geometric mean of the term immediately before and the term immediately after. Is it necessarily true that from a certain point on, the means are either all arithmetic means or all geometric means?

5. A quadrilateral shaped frame has pivots at its corners and can freely move. Show that, if its diagonals are ever at right angles, then they are always at right angles.

6. In the rhombus $ABCD$, $\angle A = 120°$. M is a point on BC and N is a point on CD such that $\angle MAN = 30°$. Prove that the circumcentre of triangle MAN lies on a diagonal of $ABCD$.

Senior Questions

1. Let a and b be arbitrary positive integers. The sequence $\{x_k\}$ is defined by $x_1 = a$, $x_2 = b$ and for $k \geq 3$, x_k is the greatest common divisor of $x_{k-1} + x_{k-2}$.

   (a) Prove that the sequence is eventually constant.

   (b) How can this constant value be determined from a and b?

2. In an arbitrary binary number, consider any digit 1 and any digit 0 which follows it, not necessarily immediately. They form an odd pair if the number of other digits in between is odd, and an even pair if this number is even. Prove that the number of even pairs is greater than or equal to the number of odd pairs.

3. There are $n \geq 4$ points on a circle, numbered 1 to n in some order. Two non-neighbouring points A and B are said to be "linked" if points on at least one of the two arcs between A and B all have numbers smaller than those of both A and B.

   Prove that the number of "linked" pairs of points is exactly $n - 3$.