Never Stand Still | Science | School of Mathematics and Statistics

MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 10, July 25, 2016

1. Since each 3×4 and 4×3 rectangle needs to have at least one black square, the minimum possible is 12. This can be achieved with the following configuration.

		X				X			X		
					X						
		X							X		
					X			X			
			X								
		X				X			X		

2. By substituting the two points $A(-2,1)$ and $B(2,9)$ into the equation of the parabola, we obtain two equation

$$4a - 2b + c = 1 \quad (1)$$
$$4a + 2b + c = 9. \quad (2)$$

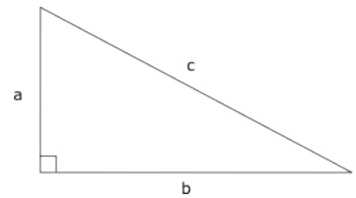
From this we get $b = 2$ and $c = 5 - 4a$. We also know that $\Delta = b^2 - 4ac < 0$ and so

$$\Delta = 4 - 4a(5 - 4a) = 16a^2 - 20a + 4 = (16a - 4)(a - 1) < 0$$

which tells us $1/4 < a < 1$. Now the x-coordinate of the vertex is $x^* = \frac{-b}{2a}$ and subbing in b and the range of a we have $-4 < x^* < -1$.

3. (a) If we choose any 3 numbers, we will either have at least two even numbers, or at least two odd numbers. If we subtract two numbers of the same parity, it is clear that their difference will be divisible by 2.

(b) Fix n. Consider the remainder of a number when we divide by $n - 1$. There are $n - 1$ possible remainders $\{0, ..., n - 2\}$. Thus, if we pick n numbers, it is clear that we must have at least 2 numbers with the same remainder. If we have two numbers with the same remainder, their difference will have remainder 0 when divided by $n - 1$ and hence will be divisible by $n - 1$.



4. Let a and b be the shorter two sides of the triangle and c be the hypotenuse. Then we have
$$\frac{1}{2}ab = 3(a+b+c).$$
Dividing both sides by 3, using $c = \sqrt{a^2+b^2}$ and rearranging
$$\frac{ab}{6} - (a+b) = \sqrt{a^2+b^2}.$$
Squaring both sides,
$$\frac{a^2b^2}{36} - \frac{ab}{3}(a+b) + (a+b)^2 = a^2+b^2.$$
Simplifying,
$$a^2b^2 - 12ab(a+b) + 72ab = 0.$$
Factoring ab,
$$ab(ab - 12(a+b) + 72) = 0.$$
We know that $ab \neq 0$ so we are left with
$$ab - 12(a+b) + 72 = 0.$$
Now using the hint we get
$$(a-12)(b-12) = 72 = 2^3.3^2.$$
By equating all possible factorisations of 72, we get 6 pairs of solutions for a and b as follows
$$(13,84), (14,48), (15,36), (16,30), (18,24), (20,21).$$

5. Using the difference of two cubes, we get
$$(x-y)(x^2+xy+y^2) = 91 = 1*91 = 7*13.$$
Also notice that $x^2+xy+y^2 = (x+y/2)^2 + 3y^2/4 \geq 0$ which also tells us $x - y \geq 0$. Since 7 and 13 are prime, by equating the factors on the LHS with ways to factorise 91 we have 4 cases:

   (a) $x - y = 91$ and $x^2+xy+y^2 = 1$
   (b) $x - y = 1$ and $x^2+xy+y^2 = 91$
   (c) $x - y = 13$ and $x^2+xy+y^2 = 13$

(d) $x - y = 7$ and $x^2 + xy + y^2 = 7$

The first and third cases have no solutions, the second case has solutions $\{x = 5, y = 6\}, \{x = -6, y = -5\}$ and fourth case has solutions $\{x = 3, y = -4\}, \{x = -4, y = 3\}$.

6.

Senior Questions

1. For $p = 2$ we have $2^2 + 2^2 = 8$ which is not prime. For $p = 3$, we have $2^3 + 3^2 = 17$ which is prime. For $p > 3$ (odd), we claim that $2^p + p^2$ is divisible by 3. Consider the remainder of $2^p = (3 - 1)^p$ when divided by 3. By considering binomial coefficients, the remainder will just be the last term $(-1)^p = -1$. Also consider the remainder of p^2 when divided by 3. If $p = 3k + 1$, then $p^2 = 9k^2 + 6k + 1$ which has remainder 1, and if $p = 3k + 2$, then $p^2 = 9k^2 + 12k + 3 + 1$ which also has remainder 1. Thus, $2^p + p^2$ is divisible by 3 for $p > 3$ and the only solution is $p = 3$.

2. Putting $y = -x$ gives us $f(-x) = -f(x)$. By putting $y = x$, we obtain $f(x^3) = xf(x^2)$. Substituting this into the original equation we get

$$xf(y^2) + yf(x^2) = (x + y)f(xy).$$

Now put $y = 1$, and $y = -1$ to get two equation, and add them to get $f(x) = cx$ where $c = f(1)$. (You can check this is indeed a solution by putting it back into the original equation.)

3. We write as $y^2 + 3x^2y^2 - 30x^2 = 517$. Then we factorise

$$(y^2 - 10)(3x^2 + 1) = 517 - 10 = 507.$$

Now $507 = 3 * 13^2$. Since x and y are integers, $3x^2 + 1$ cannot be a multiple of 3. Also, $3x^2 + 1$ cannot equal 169. Note that if $3x^2 + 1 = 1$, we also get no solution. This leaves $3x^2 + 1 = 13$ and $x = \pm 2$, and $y^2 - 10 = 39$ so $y = \pm 7$. So we have solutions $(x, y) = \{(2, 7), (2, -7), (-2, 7), (-2, -7)\}$.