UNSW AUSTRALIA

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MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 11, August 1, 2016

1. In the sum $29x + 30y + 31z = 366$, only the values of x and z affects the last digit of 366. In particular, $z = x - 4 + 10i$ for some integer i. Therefore,

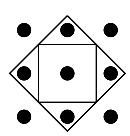
$$\begin{aligned} 29x + 30y + 31z &= 366 \\ 60x + 30y - 124 + 310i &= 366 \\ 60x + 30y + 310i &= 490, \end{aligned} \tag{1}$$

and

$$x + y + z = 2x - 4 + 10i + y. \tag{2}$$

Since we wish to minimise the sum $x+y+z$, we want to make $|i|$ as small as possible. By (1), it is clear that the smallest possible $|i|$ is $i = 1$. Therefore, (1) becomes $60x + 30y = 180$. To minises $x + y$ satisfying $60x + 30y = 180$, we take $x = 3$ and $y = 0$. Then by (2), $x + y + z = 12$.

2. Two.



3. Since

$$\begin{aligned} n^3 + 100 &= n^3 + 1000 - 900 \\ &= (n + 10)(n^2 - 10n + 100) - 900. \end{aligned}$$

For $n^3 + 100$ to be divisible by $n + 10$, $(n + 10)(n^2 - 10n + 100)$ and 900 must both be divisible by $n - 10$. The former term is always divisible by $n - 10$, while the later is divisible by any factors of 900. Since the largest factor of 900 is itself, we conclude that the largest possible n is 890.

4. Let x be such a five-digit natural number. Then x is divisible by 5 so that the last digit of x must be 5 or 0. Therefore, the sum of the first 4 digits of x must be divisible by 5, since adding 0 or 5 to any number preserves its divisibility by 5.

Next, to count the number of ways to form the first 4 digits of x. Note that there are 9 possible choices for the first digit of x (since we can not have 0 as a leading digit for a five-digit number), and 10 possible choices for the second and third digits. Furthermore, once the first 3-digits of x had been chosen, there are only 2 choices for the fourth digit such that the sum of the first 4 digits of x is either a multiple of 5 or 10. Hence, we conclude that the number of ways to choose the first 4-digits of x is $9 \times 10^2 \times 2 = 1800$.

Finally, since there are only 2 possibilities for the last digit of x, the total number of ways we can form this 5-digit natural number is $1800 \times 2 = 3600$.

5. First, we determine the area of the four non-shaded triangles. Let the edges of the diamond be the bases of these triangles, h the height and a the length of the non-base side of the triangles. Hence, trigonometric identities for the non-shade triangles, we have

$$1^2 = a^2 + a^2 - 2a\cos(120) = 3a^2.$$

Hence, $a = \frac{1}{\sqrt{3}}$. Therefore, $h = a\cos(60) = \frac{1}{2\sqrt{3}}$. So that the total area of the four non-shaded triangles is

$$4 \times \frac{1}{2} \times \frac{1}{2\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Now to find the area of the circle inscribed, we note that the radius of the circle is $r = \frac{1}{2}(1 - 2h)$. Therefore, the area of the circle is

$$\pi \times \left(\frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right)^2 = \pi\left(\frac{2-\sqrt{3}}{6}\right).$$

The area of the shaded region is therefore

$$1 - \left(\frac{1}{\sqrt{3}} + \pi\left(\frac{2-\sqrt{3}}{6}\right)\right).$$

6. We can keep as many as 2015 of this set to form the required product. Consider the product of all number from the set $1!, 2!, \ldots 2016!$,

$$\begin{aligned} 1! \times 2! \times \ldots \times 2016! &= 1^{2016} \cdot 2^{2015} \cdot 3^{2014} \cdot \ldots \cdot 2015^2 \cdot 2016^1 \\ &= 2^{2015} \cdot 4^{2013} \cdot \ldots \cdot 2016^1 \times 1^{2016} \cdot 3^{2014} \cdot \ldots \cdot 2015^2 \\ &= 2 \cdot 4 \cdot \ldots \cdot 2016 \times 2^{2014} \cdot 4^{2012} \cdot \ldots \cdot 2014^2 \times 1^{2016} \cdot 3^{2014} \cdot \ldots \cdot 2015^2 \\ &= 1008! \times 2^{1008} \times 2^{2014} \cdot 4^{2012} \cdot \ldots \cdot 2014^2 \times 1^{2016} \cdot 3^{2014} \cdot \ldots \cdot 2015^2. \end{aligned}$$

The only term that is not a perfect square on the RHS of the last displayed equation is 1008!.

Senior Questions

1. Since $b+a$ is a multiple of $b-a$, $b+a=k(b-a)$ for some integer $k>1$ ($k\neq 1$, since we are given that $a\neq 0$). Hence,

$$\frac{b}{a}=\frac{k+1}{k-1}=2+\frac{2}{k-1}\geq 3.$$

Similarly, $\frac{c}{b}\geq 3$. So that $\frac{c}{a}=\frac{c}{a}\times\frac{b}{a}\leq 9$. Now since b has 2012-digits, c must have at least 2012-digits. Moreover, a has 2011-digits and $c\leq 9a$, which implies c can not have more than 2012-digits; c has exactly 2012-digits.

2. Writing out the summation, and then splitting/rearrange terms of the summation gives

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{k}{5^k}&=\frac{1}{5^1}+\frac{2}{5^2}+\frac{3}{5^3}+\dots\\&=\frac{1}{5^1}+\frac{1}{5^2}+\dots+\frac{1}{5^2}+\frac{1}{5^3}+\dots+\frac{1}{5^3}+\frac{1}{5^4}+\dots\\&=\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\frac{1}{5^k}.\end{aligned}$$

Note that $\sum_{k=j}\frac{1}{5^k}$ is a geometric series, with $r=\frac{1}{5}$ and initial term $a=\frac{1}{5^j}$, hence

$$\sum_{k=j}\frac{1}{5^k}=\frac{1}{5^j}\left(\frac{1}{1-\frac{1}{5}}\right)=\frac{1}{5^j}\times\frac{5}{4}.$$

Therefore,

$$\sum_{k=1}^{\infty}\frac{k}{5^k}=\sum_{j=1}^{\infty}\frac{1}{5^j}\times\frac{5}{4}=\frac{1}{5}\left(\frac{1}{1-\frac{1}{5}}\right)\times\frac{5}{4}=\frac{1}{5}.$$

3. Let AC and BD intersect at O. Suppose the diagonals are not perpendicular to each other. By symmetry, we may assume that $\angle AOB=\angle COD<90°$. Then by Pythagoras

$$(|OA|^2+|OB|^2)+(|OC|^2+|OD|^2)>|AB|^2+|CD|^2=221$$

while

$$(|OD|^2+|OA|^2)+(|OB|^2+|OC|^2)<|DA|^2+|BC|^2=221.$$

This is a contradiction. Hence both angles between the diagonals are 90°.