MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 12, August 8, 2016

1. Since $1 + 2 + 3 + \ldots + 99 = 4950$ (sum of an arithmetic series), we have

$$n + 2n + 3n + \ldots + 99n = n\,(1 + 2 + 3 + \ldots + 99) = 4950n.$$

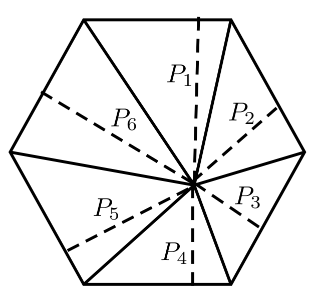
Furthermore, we can factor $4950 = 5^2 \times 3^2 \times 2 \times 11$. Therefore, for $4950n$ to be a perfect square, n must be positive and a multiple of both 11 and 2; the smallest possible n is 22.

2. Since $1 + 2 + 3 + \ldots + n$ is the sum of an arithmetic series, with common difference 1, we have

$$f(n) = \frac{\frac{n}{2}(2 + (n-1))}{n} = 1 + \frac{1}{2}(n-1).$$

Therefore $f(1) + f(2) + \ldots + f(100)$ is sum of an arithmetic series, with initial term 1 and common difference 0.5. Hence

$$f(1) + f(2) + \ldots + f(100) = \frac{100}{2}\,(2 + 99 \times 0.5) = 2575.$$



3. Let $P_1, P_2, \ldots, P_n$ be the sides of the polygon, where n is some nonnegative integer. Then for $1 \leq i \leq n$, each P_i is the height of a triangle form by the point P and two adjacent corners of the polygon; for example, as shown in the figure above for the case $n = 6$. In particular, the area of the polygon is given by

$$\frac{1}{2}d(P_1 + P_2 + \ldots + P_n), \tag{1}$$

where d is the length of the polygon. Since the area of the polygon is independent of the position of P, we conclude by (1) that $P_1 + P_2 + \ldots + P_n$ is the same for all positions of P.

4. Multiplying both sides of the given equation by $(x-2)(x-3)(x-5)$ gives

$$x^2 - p = A(x-3)(x-5) + B(x-2)(x-5) + C(x-2)(x-3). \tag{2}$$

If we substitute $x = 2$ into (2), then $4 - p = 3A$. Similarly, substituting $x = 3, 5$ into (2) yields $9 - p = -2B$ and $25 - p = 6C$. Therefore, we have the system of equations

$$\begin{aligned} p &= 4 - 3A \\ p &= 9 + 2B \\ p &= 25 - 6C. \end{aligned}$$

The smallest possible p is 7, with $A = B = -1$ and $C = 3$.

5. We call any 1 cm sides of the lattice an edge, and any point of intersection between edges a vertex. We denote the degree of a vertex by the number of edges incident on that vertex. Consider a vertex with degree 3: since there is an odd number of edges attached to this vertex. Thus, if we can not have multiple threads on an edge, then at least one end of a thread must start at this vertex. In particular, for our 4 cm by 4 cm square lattice, there are 12 vertices with degree 3. Hence, if we only have a total of 40 cm of threads (so that threads can not pass over an edge more than once), then there must be at least $12/2 = 6$ pieces of thread to fill out the lattice.

   (a) No possible, too few threads.

   (b) It is possible.

6. Since $(x+y)^2 = x^2 + 2xy + y^2$. If we set $x = 2^{2016}$ and $y = 3^{2016}$, then

$$\begin{aligned} \left(2^{2016} + 3^{2016}\right)^2 &= \left(2^{2016}\right)^2 + 2 \times 2^{2016} \times 3^{2016} + \left(3^{2016}\right)^2 \\ &= 4^{2016} + 2 \times 6^{2016} + 9^{2016}. \end{aligned}$$

Therefore,

$$\sqrt{4^{2016} + 2 \times 6^{2016} + 9^{2016}} = 2^{2016} + 3^{2016}. \tag{3}$$

Due to (3), it remains to find the last two digits of 2^{2016} and 3^{2016}.

To find the last two digits of 2^{2016}, we search for the smallest possible integer n, such that the last two digits of (16^{i+nk}) coincides with the last two digits of 16^i for all integer $i, k \geq 1$. To do this, we write

$$\begin{aligned} 16^{i+1} &= 16 \times 16^i \\ &= (10 \times 16^i + 5 \times 16^i + 16^i) \\ &= (10 \times 16^i + 80 \times 16^{i-1} + 16^i). \end{aligned}$$

From the above displayed equation, it can be seen that the last two digits of 10×16^i is always 60, and the last two digits of $80 \times 16^{i-1}$ is always 80. Therefore, the last digit of 16^{i+1} coincides with 16^i, and the second last digit of 16^{i+1} is increased by 4 compared to 16^i. In particular, the last two digits of 16^{i+5} coincides with 16^i. Hence, $n = 5$. Thus, the last two digits of

$$2^{2016} = 16^{504} = 16^{4+(5)(100)},$$

is 36.

By similar constructions, we can find the last two digits of 3^{2016}, which is 21. Hence, the required number is 57.

Senior Questions

1. $a = 456$, $b = 546$ and $c = 1554$.

2. if $5a^2 - 7b^2 = 9$ then 5 does not divide b hence the remainder on dividing b by 5 is $1, 2, 3$ or 4; i.e $b = 5c + d$, $d = 1, 2, 3$ or 4. Therefore

$$b^2 = (25c^2 + 10cd) + d^2,$$

with $d^2 = 1, 4, 9$ or 16. Hence

$$9 = 5a^2 - 7b^2 = 5(a^2 - 35c^2 - 14cd) - e,$$

where $e = 7, 28, 63$ or 112. In particular, $5(a^2 - 35c^2 - 14cd)$ is equal to $16, 27, 72$ or 121, which is impossible. Thus, no such b exists.

3. A convex 8 sided polygon with all angles equal has angles $180 - \frac{360}{8} = 135°$. Hence the polygon can be fitted into a rectangle as shown. Let the lengths of the sides be a, b, c, d, e, f as shown all integers. Now in $45°, 45°, 90°$ triangle the ratio of the sides are $1 : 1 : \sqrt{2}$. Hence $a = \sqrt{2}x$, $c = \sqrt{2}y$, $f = \sqrt{2}v$, $d = \sqrt{2}u$.

$$\begin{aligned} u + e + v &= x + b + y \\ e - b &= x + y - u - v = \frac{a + c - f - d}{\sqrt{2}} \\ \sqrt{2} &= \frac{a + c - f - d}{e - b} \quad e \neq b \\ &= \frac{p}{q} \quad p, q \text{ positive integers.} \end{aligned}$$

This says that $\sqrt{2}$ is rational which is not true. Hence there is no such octagon.