1. Consider
\[ f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \ldots = 1 + x + x^2 + x^3 + x^4 \ldots + \frac{1}{1 - x}, \]
where last line is due to the sum of an infinite geometric sequence. Hence, setting \( x = \frac{1}{2^2} \) in \( f(x) \), we have
\[ f\left(\frac{1}{2^2}\right) = \left(1 \frac{1}{2}\right) \left(1 \frac{1}{2^2}\right) \left(1 \frac{1}{2^4}\right) \left(1 \frac{1}{2^8}\right) \ldots = \frac{1}{1 - \frac{1}{2^2}} = \frac{4}{3}. \]

2. Let \( x \) be the different of their ages in days. When Alice was twice as old as Bert was, their ages are \( 2x \) and \( x \). When Bert’s age was \( 2 \times (2x) \), Alice’s age was \( 5x \). In another 1296 days Bert’s age will be \( 2 \times (5x) = 10x \), and Alice’s age will be \( 11x \). Therefore,

\[ (10x - 1296) + (11x - 1296) = 11016. \]

Solving the above equation gives \( x = 648 \).

So Alice’s age is \( 11x - 1296 = 5832 \) and Bert’s age is \( 10x - 1296 = 5184 \).

3. Note that \( f(x) \geq 2 \) is equivalent to \( x + \frac{1}{x} - 2 \geq 0 \). We have
\[ x + \frac{1}{x} - 2 = \frac{1}{x} \left(x^2 + 1 - 2x\right) = \frac{1}{x} (x - 1)^2 \geq 0, \]
where we have used the fact that \((x - 1)^2 \geq 0\), and assumption that \( x > 0 \) to obtain the last line.
4. Since \(2^x = 6^{-z}\), we have
\[
2 = 6^{-\frac{z}{x}}. \tag{1}
\]
Similarly, since \(3^y = 6^{-z}\), we have
\[
3 = 6^{-\frac{z}{y}}. \tag{2}
\]
Therefore, combining \((1)\) and \((2)\), we have
\[
6 = 2 \times 3 = 6^{-\frac{z}{x}} \times 6^{-\frac{z}{y}} = 6^{-\frac{z}{x} - \frac{z}{y}}.
\]
In particular,
\[
1 = -\frac{z}{x} - \frac{z}{y},
\]
so that
\[
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.
\]

5. The solution is 89. This can be obtain by using binomial expansion carefully.

Alternatively, note that
\[
\left(\frac{1 + \sqrt{5}}{2}\right)^{11} + \left(\frac{1 - \sqrt{5}}{2}\right)^{11}
\]
is the 11th term of the Fibonacci number, see \(\text{https://en.wikipedia.org/wiki/Fibonacci_number}\) or Question sheet 6, 2016.

6. Let \(x\) the number of dollars and \(y\) the number of cents on the cheque. Note that three times the value of the cheque must be less than \$100.22, which implies \(x < 34\). Now, we can write the value of the cheque as \(100x + y\) cents, then the amount the bankers gave out was \(3(100x + y) - 22\) cents. Therefore,
\[
100y + x = 3(100x + y) - 22
\]
\[
97y = 299x - 22
\]
\[
97(y - 3x) = 8x - 22.
\]
Hence, using \(x < 34\)
\[
97(y - 3x) = 8x - 22 \leq 250. \tag{3}
\]
The LHS equality of \((3)\) implies \(y - 3x\) must be even. The RHS inequality implies \(y - 3x \leq 2\). From this, we conclude that
\[
y - 3x = 2
\]
\[
97 \times 2 = 8x - 22.
\]
Solving the above system simultaneously yields \(x = 87\) and \(y = 27\).
Senior Questions

1. Let \( f(x) \) denote the number of consecutive primes between \( x \) and \( x + 2015 \). Clearly \( f(1) > 15 \). Moreover, for consecutive inputs \( x \) and \( x + 1 \), the function \( f \) can only vary by 0, 1 or \(-1\); i.e \( f(x) \) differs to \( f(x + 1) \) at most \( \pm 1 \). Hence, if we can find an integer \( n \), such that \( f(n) = 0 \), then since \( f(1) > 15 \) and given how “smoothly” \( f \) varies, there exist an integer \( 1 < m < n \) such that \( f(m) = 15 \).

The integer \( n \) exist: we can find it directly, as \( n = 2016! + 2 \), implies \( f(n) = 0 \).

2. Suppose \( n \) exist, then there is a prime number \( p \), such

\[
\begin{align*}
n^3 - 9n + 27 &= 81p \\
n^3 &= 81p + 9n - 27 \\
&= 9(9p + n - 3).
\end{align*}
\]

Hence, \( n^3 \) is divisible by 9 which implies \( n \) is divisible by 3. Therefore, there is an integer \( k \), such that \( n = 3k \). But then

\[
n^3 - 9n + 27 = 27k^3 - 27k + 27 = 27(k(k - 1)(k + 1) + 1),
\]

which is not divisible by 81, since \( k(k - 1)(k + 1) \) is always divisible by 6. This is a contradiction.

3. Label the shade quadrilateral by \( WZYX \), and let \( T \) be a point external to \( ADCB \) such that \( ATZW \) is a parallelogram; as shown above. It is straight forward to show that \( WZYX \) forms a parallelogram, and \( WZYX \) is congruent to \( ATZW \).

Note that \( \triangle ADW \) and \( \triangle GDZ \) are similar. Hence, if the area of \( \triangle GDZ \) is \( x \), then the area of \( \triangle ADW \) is 4\( x \). Thus, the area of the quadrilateral \( AGZW \) is 3\( x \). Moreover, \( \triangle ATG \) is congruent to \( \triangle GDZ \). Hence, the area of \( \triangle ATG \) is \( x \). Thus, the area of the parallelogram \( ATZW \) is 4\( x \). Since \( WZYX \) is congruent to \( ATZW \), it follows that the shaded region is 4\( x \).

Now suppose the area of \( \triangle AWH \) is \( y \), then by similar arguments as before, the area of \( WZYX \) is 4\( y \). Hence, \( x = y \).

Finally by symmetry, the area of \( \triangle ADW \) is equal to the area of \( \triangle YCB \), and the area of \( \triangle AXB \) is equal to the area of \( \triangle DCZ \). In particular, each triangle \( \triangle ADW \), \( \triangle YCB \), \( \triangle AXB \) and \( \triangle DCZ \) have area 4\( x \), and \( WZYX \) have area 4\( x \). It follows that since the area of \( ADCB \) is 1, the area of \( WZYX \) is \( \frac{1}{20} \times 4 = \frac{1}{5} \).