MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 4, May 22, 2016

1. Let $x = \sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+...}}}}$, then

$$\begin{aligned} x^3 &= 6+\sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+...}}} \\ &= 6+x \\ x^3 - x - 6 &= 0. \end{aligned}$$

   Hence, $x = 2$.

2. (a) Let $x = d_1 d_2 \ldots d_n$ be a n-digits long integer, where $d_1, d_2, \ldots d_n$ are its n-digits; That is,
$$x = d_1 + 10 \times d_2 + \ldots + 10^{n-1} \times d_n.$$
   Then
$$x = d_1 + d_2 + \ldots + d_n + (10-1) \times d_2 + (10^2 - 1) \times d_3 + \ldots + (10^{n-1} - 1) \times d_n. \quad (1)$$
   Note that on the RHS of (1), the factors $10^i - 1$ are divisible by 9, for all $1 \leq i \leq n-1$. Therefore, x is divisible by 9 if and only if $d_1 + d_2 + \ldots + d_n$ is divisible by 9.

   (b) We claim that in a based b-system, an integer $x = d_1 d_2 \ldots d_n$ is divisible by $b-1$ if and only if the sum of its digits is divisible by $b-1$. By a similar argument as in (1), one has
$$x = d_1 + d_2 + \ldots + d_n + (b-1) \times d_2 + (b^2 - 1) \times d_3 + \ldots + (b^{n-1} - 1) \times d_n.$$
   Since $b^i - 1$ are divisible by $b-1$, for $1 \leq i \leq n-1$ (i.e by polynomial long division argument, see `https://en.wikipedia.org/wiki/Polynomial_long_division`), the result follows from part (a).

3. First, note that $1 < \sqrt{3-x} - \sqrt{x+1}$ implies $-1 \leq x \leq 3$. Now,

$$\begin{aligned} 1 &< \sqrt{3-x} - \sqrt{x+1} \\ &< \left(\sqrt{3-x} - \sqrt{x+1}\right)^2 \\ &= (3-x) - 2\sqrt{(3-x)(x+1)} + (x+1) \\ &= 4 - 2\sqrt{(3-x)(x+1)}, \end{aligned}$$

which implies

$$\begin{aligned}2\sqrt{(3-x)(x+1)} &< 3\\ 4(3-x)(x+1) &< 9\\ -4x^2+8x+3 &< 0.\end{aligned}$$

Solving the above quadratic inequality and recalling that $-1 \leq x \leq 3$ gives $1 \leq x < -1 + \frac{1}{2}\sqrt{7}$.

4. Consider $g(x) := f(x) - 1$. We have

$$g(xy) = f(xy) - 1 = yf(x) + xf(y) - x - y = yg(x) + xg(y).$$

Next, set $h(x) = \frac{g(x)}{x}$ then

$$h(xy) = \frac{g(xy)}{xy} = \frac{g(x)}{x} + \frac{g(y)}{y} = h(x) + h(y).$$

Hence, $h(x) = \ln(x)$. Thus, $f(x) = x\ln(x) + 1$.

5. (a) Let x and y be the two natural numbers. We wish to find the largest value of $xy = x(2016 - x)$, the concaved down parabola with roots 0 and 2016. The maximum is at the turning point $x = 1008$. So the greatest product is $xy = 1008^2$.

   (b) Let $x_1 \leq x_2 \leq \ldots \leq x_n$ be the n natural numbers. If any of the natural numbers is 1, then $x_1 = 1$ and the product $x_1x_2\ldots x_n$ is not maximum, because we can add x_1 to any one of the other natural numbers and always end up with a greater product.

   Now, suppose that one of natural number say x_1 is greater than 3, then we can split x_1 into $x_1 - 2$ and 2. The result of the product is

$$2 \times (x_1 - 2) \times x_2x_3x_4\ldots x_n \geq x_1x_2\ldots x_n.$$

   Therefore, the product $x_1x_2\ldots x_n$ is greatest when each $x_1, x_2, \ldots x_n$ either 2 or 3.

   Finally, if three or more of the natural number are 2's, then we can combine them into two 3's, then product will be greater than it was before since $2\times2\times2 < 3\times3$.

   Hence, the to obtain the greatest product of $x_1, x_2, \ldots x_n$ when $x_1 = x_2 = \ldots = x_n = 3$. In particular, the greatest product of the natural numbers that sum to 2016 is $3^{2016/3}$.

6. The sum of angles of a polygon is $(n-2)\times 180$ (you may want to show this). Hence, the internal angles of each pentagon and decagon is $108°$ and $144°$ respectively. Since the length of the sides of both shapes are 1, to make sure there is no gap between tiles, we must join two corners of the a pentagon to each one corner of the decagon to make $2 \times 108° + 144° = 360°$; This will not work without overlaps.

Senior Questions

1. Let $x = 2^a$, $y = 2^b$ and $z = 2^c$ be the solution of the said problem. One has

$$2^{7a} + 2^{9b} = 2^{8c}$$
$$2^{7a}(1 + 2^{9b-7c}) = 2^{8c},$$

which implies $9b = 7c$, otherwise $1 + 2^{9b-7c}$ will be either odd or a fraction. Moreover,

$$2^{7a} + 2^{9b} = 2^{8c}$$
$$2^{7a+1} = 2^{8c},$$

since $7a = 9b$. Therefore, $7a + 1 = 8c$. Hence, we can write the solution as

$$a = t,$$
$$b = \frac{7}{9}t,$$
$$c = \frac{7t+1}{8},$$

for some real number t. However, we require integer solutions for a, b and c. Since $a = t$, we must limit t to the set of non-negative integer. Moreover, b and c both needs to be non-negative integers, so we further restrict t to be a subset of the non-negative integers say $t = \alpha k + \beta$, for some positive integers α, β, k. Clearly, α and β both must be a multiple of 9 so that $b = \frac{7}{9}k$ is an integer. Also, 7α and $7\beta + 1$ both must be a multiple of 8 for $c = \frac{7t+1}{8}$ to be an integer. Since 8 and 9 are co-prime, we set $\alpha = 8 \times 9 = 72$ and $\beta = 9$. Hence, the integral solutions are

$$a = 72k + 9,$$
$$b = 56k + 63 = 56(k + 1) + 7,$$
$$c = 63k + 8,$$

for some positive integer k. Therefore, we conclude that there are infinitely many solutions to this problem.

2. Let the first term and the common difference of the arithmetic progression be a and $d > 0$ respectively. Let the first term and the common ratio of the geometric progression be b and $r > 1$ respectively. Then $b = a + id$, $br = a + jd$ and $br2 = a + kd$ for some integers i, j and k such that $0i < j < k$. It follows that $b(r - 1) = (j - i)d$ and $br(r - 1) = (k - j)d$, so that $r = \frac{k-j}{j-i}$ is a rational number. Let $t = ad$. From $a + jd = br = r(a + id)$, we have $t + j = rt + ri$. Hence $t = \frac{j-ri}{r-1}$ is also rational. Divide all the terms of both progressions by d. Then the arithmetic progression has first term t and common difference 1 while the geometric progression has first term bd and common ratio r. Let $t = \frac{p}{q}$ where p and q are relatively prime positive integers. Then all terms in the arithmetic progression are of the form $\frac{p+kq}{q}$ for some non-negative integer k. If r is not an integer, then when n is a sufficiently large positive integer, the expression of $\frac{b}{d}r^n$ as a fraction in the simplest terms will have a denominator greater than q. This contradicts the hypothesis that every term of the geometric progression is a term of the arithmetic progression.

3.