MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 5, May 30, 2016

1. By Pythagoras, $c^2 = a^2 + b^2$. Hence, $a^2 = c^2 - b^2 = (c-b)(c+b)$. So that for $a^2 = b+c$, we must have $b - c = 1$. Therefore, $a^2 = b + c = 2b + 1$, which implies a is odd because b is an integer. Let k be an integer, then a must be in the form $a = 2k + 1$.

   Hence $a = 2k + 1$, $b = (2k + 1)^2 = 4k^2 + 4k + 1$ and $c = 4k^2 + 4k + 2$, for $k = 1, 2, 3, \ldots$ are the solutions.

2. Let $x = d_0 d_1 d_2 \ldots d_{n-1} d_n$. Then we can split x into the sum of two numbers, one consist of the odd digit the other the even digits; That is

$$\begin{aligned} x &= 10^n \times d_0 + 10^{n-1} \times d_1 + 10^{n-2} \times d_2 + \ldots + 10 \times d_{n-1} + d_n \\ &= (10^n \times d_0 + 10^{n-2} \times d_2 + \ldots) + (10^{n-1} \times d_1 + 10^{n-3} d_3 \ldots). \end{aligned}$$

   Now, the remainder of 10^k divided by 11 is 1 when k is odd, and -1 when k is even. Recall the properties of remainders `https://en.wikipedia.org/wiki/Modular_arithmetic`. The remainder of $10^n \times d_0$ divide by 11 is either d_0 or $-d_0$ depending on whether n is even or odd. Therefore, the remainder of $(10^n \times d_0 + 10^{n-2} \times d_2 + \ldots)$ divided by 11 is either $d_0 + d_3 + \ldots + d_n$ or $-(d_0 + d_3 + \ldots + d_n)$ depending on whether n is even or odd. Similarly, the remainder of $(10^{n-1} \times d_1 + 10^{n-3} d_3 \ldots)$ divided by 11 is either $d_0 + d_3 + \ldots + d_n$ or $-(d_0 + d_3 + \ldots + d_n)$ depending on whether n is even or odd. Therefore, we can conclude that the remainder of x divided by 11 is $\pm[(d_0 + d_2 + \ldots) - (d_1 + d_3 + \ldots)] = \pm[a - b]$. Thus, if $a - b$ is divisible by 11 then so is x.

3. Since $x_1, x_2, \ldots, x_n$ can only be $+1$ or -1, we can find what x is by just counting the number of -1 values in the set $\{x_1, x_2, \ldots, x_n\}$. Let x_k denote the value of x, when the number of -1 values from the set $\{x_1, x_2, \ldots, x_n\}$ is equal to k. Then,

$$x_k = -k + (n - k) + (-1)^k.$$

   Suppose n is even. If k is even, then $x_{2m} = -2m + (n - 2m) - 1 = n - 4m + 1$ for $m = 1, 2, \ldots, n/2$. If k is odd, then $x_{2m+1} = n - 4m + 1$ for $m = 0, 1, 2, \ldots, n/2 - 1$. Hence, $x_{2m+1} = x_{2m}$, for all $m = 1, 2, \ldots n/2 - 1$. Therefore, the number of unique x_n values is equal to $n/2 + 1$.

   If n is odd, the by similar arguments, the number of distinct values of x_n is $(n+1)/2+1$.

4.

5. Apply the change of coordinates $X = x/10$ and $Y = 10y$. Then $X = 10\cos(10Y$ and $Y = 10\cos 10X$. In particular, the graph of X and Y is symmetric. Let A be the sum of their X-coordinate, and B be the sum of their Y-coordinate. By the symmetry of graph of X and Y, one has $\frac{A}{B} = 1$. Moreover, by definition one has $A = a/10$ and $B = 10b$. Hence, $\frac{a}{b} = \frac{10A}{B \div 10} = 100$.

6. Label the points $p_1, p_2, \ldots, p_{100}$. Draw the $p_1, \ldots, p_{99}$ evenly spaced on a circle in order, and then place the p_{100} in the center of the circle. Suppose we are able to draw 50 line segments each intersect one another. Then by construction, no lines can pass over more than 2 points. Hence, we may assume without loss of generality, that the points are connect in pairs, and that p_1 is connected to p_{100}. Consider the point p_{50}, if it is connected to p_k for $1 < k < 50$, then the line $p_k p_{50}$ can not possibly intersect $p_1 p_{100}$ because they belong to different halves of the circle (separated by the diameter pass through p_{50}). If p_{50} is connected to p_k for $50 < k < 100$, then again the lines $p_k p_{50}$ and $p_1 p_{100}$ belows to different halves of the circle (separated by the diameter pass through p_{49}).

7. For $x^x + 1$ to be divisible by 2^n, $x^x + 1$ must be even, which implies x must be odd. Now by using polynomial division argument `https://en.wikipedia.org/wiki/Polynomial_long_division`, one can show that

$$x^x + 1 = (x+1)(x^{x-1} - x^{x-2} + x^{x-3} - \ldots + x^2 - x + 1).$$

   Since the term $(x^{x-1} - x^{x-2} + x^{x-3} - \ldots + x^2 - x + 1)$ is the sum of odd number of odd numbers, it is an odd number, and therefore can not be divided by 2^n. It follows that $x + 1$ must be divisible by 2^n; that is x must be a multiple of $2^n - 1$, so that the least value of x for which $x^x + 1$ is divisible by 2^n is $2^n - 1$.

Senior Questions

1. Let the number we are attempting to find be n. If we add the digits of n, we get $1+2+\ldots+8+9 = 45$. Recall that an integer n is divisible by 9 if and only if the sum of its digits is divisible by 9. Since 9 divides 45, the number n is always divisible by 9. Thus, our problem is reduce to finding n, such that n is divisible by $99 \div 9 = 11$. Note that we have a divisibility by 11 rule from Q2, so to complete this problem, we just need to arrange the digits of n, to find the smallest possible combination, such that the sum of the odd digits a and the sum of the even digits b of n satisfies $a - b = 0 \mod 11$.

2. Since a, b, c, d, e are consecutive positive integers, $a = c-2, b = c-1, d = c+1$ and $e = c+2$. So that $a+b+c+d+e = 5c = x^3$, for some integer x. Also, $b+c+d = 3c = y^2$, where y is some positive integer. Since $5c = x^3$, and c is an integer, x must be a multiple of 5. Hence $c = 25m^3$ for some integer m. Now, we know that $c < 10,000$, $m^3 < 400$, $m \leq 7$. Since there is only 7 cases for m, we can easily test them to see which one also satisfies $3c = y^2$.The only solution is $m = 3$; that is $c = 675$.

3. Let $y = a + b$, where $a = \sqrt[3]{x + \sqrt{x^2 + 1}}$ and $b = \sqrt[3]{x - \sqrt{x^2 + 1}}$. Therefore, we wish to find values of x much that y is an integer. Note that $a^3 = x + \sqrt{x^2 + 1}$, $b^3 = x - \sqrt{x^2 + 1}$. Hence $a^3 + b^3 = 2x$ and $a^3 b^3 = -1$; $ab = -1$. Which implies

$$\begin{aligned} y^3 &= (a + b)^3 \\ &= a^3 + 3a^b + 3ab^2 + b^3 \\ &= 2x - 3(a + b) \\ &= 2x - 3y. \end{aligned}$$

There, $x = \frac{1}{2}(y^3 + y)$ for all integers y.