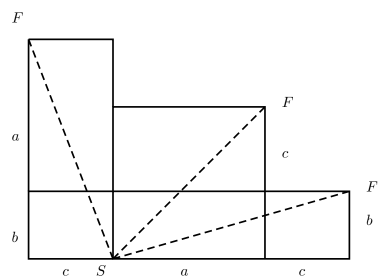
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 6, June 5, 2016

1. If we flatten out the cuboid, we can see more clearly the paths along the walls the spider can take to get to the fly. Since the shortest distance between two points is always a straight line (consequence of the triangles inequality), we see that there are three potential ways the spider can get to the fly; as shown below. The distance the spider travels in each way is



$$d_1 = \sqrt{c^2 + (a+b)^2}$$
$$d_2 = \sqrt{a^2 + (b+c)^2}$$
$$d_3 = \sqrt{b^2 + (a+c)^2}.$$

With out loss of generality, we can assume that $a > c > b$. Then d_2 is the shortest, since $d_2 = \sqrt{a^2 + (b+c)^2} = \sqrt{a^2 + b^2 + c^2 + 2bc}$, and $bc < ac < ab$.

2. First we work out how many strictly increasing numbers there are. If we list the digits $1, 2, 3, \ldots, 9$ in order, then we can make a strictly increasing number by selecting one or more digits from the list. There are $2^9 - 1$ ways to do this, since we can choose to select each of the digit or not, but we must at least select one digit.

   Next we work out how many 5-digits long strictly increasing numbers there are. We can think of this as selecting 5 digits from $1, 2, \ldots, 9$: There are $9\times 8\times 7\times 6\times 5$ ways to do this, but we don't care about the order in which we select out the 5 digits because they are already list in increasing order. Thus, there are $\binom{9}{5} = \frac{9!}{5!\times 4!}$ ways to do this.

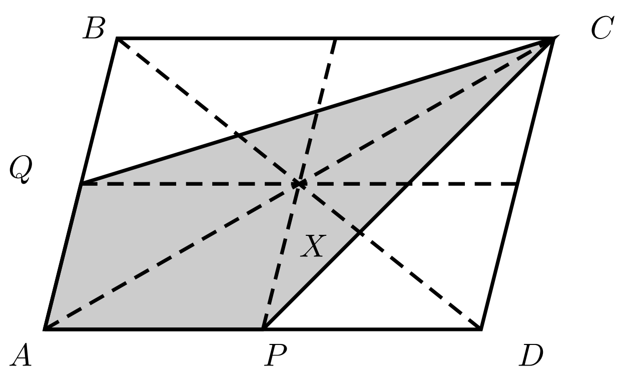
   Therefore, the answer is $18/73$.

3. If the white king is on a corners, then there are 3 spots where the black knight can not be on. If the white king is on an edge of the board but not a corner, then there are 5 spots where the black king can not be on. If the white king is not on an edge or a corner of the board, then there are 9 spots where the black king can not be on. Since there are 4 corners, $4 \times (8-2)$ edges but not corners, and 6×6 not edge or corner on a chess board, we can conclude that there are $4 \times 61 + 4 \times (8-2) \times 59 + 6 \times 6 \times 55$ ways we can do this.

4. One way to do this is to use the properties of remainders. Note that the remainder of 6^n divided by 7 is $(-1)^n$, and the remainder of 8^n divided by 7 is 1^n. Thus, $6^n + 8^n$ has remainder $(-1)^n + 1^n$ when divided by 7. Therefore, $6^n + 8^n$ is divisible by 7 if and if n is odd.

   The other way to do this is the consider the binomial expansion of $(7-1)^n$ and $(7+1)^n$.

5. Consider the figure below: Since $ABCD$ is a parallelogram, the area of the triangle



   $\triangle ABD$ is equal to half of the area of $ABCD$. Hence, if we can show that area of the shade region is equal to the area of $\triangle ABBD$, then we are done.

   Consider the triangles QCX and QBX, both triangles have the same height because the lines QX and BC are parallel. Thus, these triangles also have the same area. Similarly, the triangles PXC and PXD have the same area. Therefore, the shaded region has the same area as the triangle ABD.

6. If x is an integer, then so is y. Therefore, $[x] = x$ and $[y] = y$, so that $xy = x + y$. The solution for this case is $x = y = 2$.

   Suppose x and y are both not integers. Let $a = [x]$ and $b = [x][y] = x + y$. Then since b is an integer and $b = x + y$, we have $[y] = [b - x] = b - ([x] + 1) = b - a - 1$, which implies

$$b = [x][y] = a(b - a - 1). \tag{1}$$

   Since a and b are both integers, we can deduce from (1) that b is a factor of a. Hence, we can write (1) as

$$ac = a(ac - a - 1), \tag{2}$$

   for some integer c. If $a = 0$, then $b = a[y] = 0$, so that $b = x + y = 0$ which is impossible. So from (2), we have $ac - a - 1 = c$. Consider

$$(a-1)(c-1) = ac - a - c + 1 = 2, \tag{3}$$

where we have used the fact that $ac - a - 1 = c$ on the second equality. From (2), one has $c = 3$, $a = 2$ or $a = 3, c = 2$. Moreover, from (1), if $a = 2, 3$ then $b = 6$. Hence, $2 \leq x < 4$ and $y = b - x = 6 - x$. But x and y are both not integers in this case, so we eliminate $x = 4$ from our solution.

Thus, we have three possible solutions: $2 < x < 3$, $y = 6 - x$ or $3 < x < 4$, $y = 6 - x$ or $x = y = 2$.

Senior Questions

1. Let w_{ij} be the result of the game between the i-th and j-th players:

$$s_{ij} = \begin{cases} +1 & \text{if player } i\text{-th wins,} \\ -1 & \text{if player } j\text{-th wins,} \\ 0 & \text{if they have a draw or } ij. \end{cases}.$$

Let X_j be the total score of player j at the end of the tournament. Then the number

$$\sum_{j=1}^{n} s_{ij} X_j,$$

is the difference between the total score of those players who beats player i and those who were beaten by player i, for some fixed $1 \leq i \leq n$. Thus, the question is asking whether $\sum_{j=1}^{n} s_{ij} X_j > 0$ for every i. Consider

$$\sum_{i,j=1}^{n} s_{ij} X_j X_i. \tag{4}$$

The summation (4) is equal to 0, since $s_{ij} = -s_{ji}$. But $X_i \geq 0$ for each i, therefore (4) implies $\sum_{j=1}^{n} s_{ij} X_j \not> 0$. So, it is impossible.

2. Let us arrange all the students in the school according to the number of "A" marks they received. So, $A_1 \geq A_2 \geq \ldots \geq A_n$ where A_j is the number of "A" received by j-th student, $1 \leq j \leq n$, $A_j \geq 0$ and $\sum_{j=1}^{n} A_j = A$ where A is a total number of "A marks.

Now let us consider the first five students. According to the condition, one student (who has to be on top of the list) got at least 80% of "A" marks received by this group, which leaves no more than 20% of "A" marks remaining for the other four students. So, $A_2 + A_3 + A_4 + A_5 \leq \frac{1}{4} A_1$, and we have an estimate $A_2 \leq \frac{1}{4} A_1$. Considering students from k-th to $k+4$-th ($k + 4 \leq n$), we conclude that $A_{k+1} \leq \frac{1}{4} A_k$, which implies that $A_{k+1} \leq \frac{1}{4^k} A_1$ ($k \leq n - 5$) and $A_{n-3} + A_{n-2} + A_{n-1} + A_n \leq \frac{1}{4} A_{n-4}$. Now we have

$$\begin{aligned} A &= A_1 + A_2 + \ldots + A_{n-4} + (A_{n-3} + \ldots + A_n) \\ &\leq A_1 + \frac{1}{4} A_1 + \frac{1}{4^2} A_1 + \ldots + \frac{1}{4^{n-5}} A_1 + \frac{1}{4^{n-4}} A_1 \\ &< \sum_{k=0}^{\infty} \frac{1}{4^k} A_1 = \frac{A_1}{1 - \frac{1}{4}} = \frac{4}{3} A_1 \end{aligned}$$

Therefore, $A_1 > \frac{3}{4} A$.

3. One ways is to used induction, the identity $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$ may be helpful.

   The other way is the treat it as a first order difference equation.