MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 8, June 20, 2016

1. There are ${}^{12}C_3 = \frac{12\times11\times10}{3\times2} = 220$ ways to choose 3 Tim-Tams randomly from the 12 Tim-Tams in the box. Therefore, the chance for one person to choose 3 white Tim-Tams is 1 out of the 220. Hence, for exactly one of four person to end up with 3 white Tim-Tams is $4 \times \frac{1}{220} = \frac{1}{55}$.

2. It is possible to get it as small as the integer 2. Here is one way to play the game to get to 2. First take 2014 and 2016, then we delete them from the board and add in 2015. Next, we take the two 2015 (one from the start and one obtained from the previous step), then we delete them from the board and add on a single 2015. From this point and onwards, we can delete the largest two integers, which is always in the form n and $n-2$, for some integer $n < 2016$, and add in their average $(n-1)$. In this way, we will eventually get to: $5, 3, 2, 1$, then $4, 4, 2, 1$, then $3, 2, 1$, then $2, 2$ then 2.

3. (a) We can employ the divisibility by 9 rule to solve this. Recall that an integer is divisible by 9 if and only if the sum of its digits is divisible by 9. The numbers $18, 108, 1008, \ldots$ all have sum of digits equal to 9. Thus each $18, 108, 1008, \ldots$ is divisible by 9. Moreover, each $18, 108, 1008, \ldots$ is an even integer, and thus is divisible by 2. Therefore, every term in the sequence $18, 108, 1008, \ldots$ is divisible by 18.

   (b) We can also solve this equation with mathematical induction. One can represent the terms of the infinite sequence as $a_n = 10^n + 8$. Then clearly $a_1 = 10 + 8 = 18$ is divisible by 18. Assume that a_k is divisible by 18 for all $k \leq n$. One has

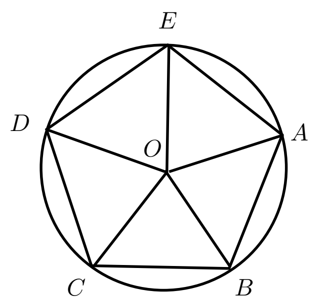
   $$a_{n+1} = 10^{n+1} + 8 = 9 \times 10^n + 10^n + 8 = 9 \times 10^n + a_n. \tag{1}$$

   The first term of (1) is divisible by 18, since 9×10^n is divisible by 9 and 2 for $n \geq 1$. The second term of (1) is divisible by 18 by induction hypothesis. Thus, a_n is divisible by 18 for all n by induction.

4. Since $40! := 40 \times 39 \times 38 \times \ldots \times 2 \times 1$, the cube of any prime number greater than 13 will be greater than 40!, thus can not possible form a cubic factor of 40!. Consider the prime factors between 1 to 13; which are $2, 3, 5, 7, 11$ and 13. We can form any prime number less or equal to 13 or any non-prime numbers less than or equal to 40 from these prime factors. Hence, all cube factors of 40! can be formed from the cubes

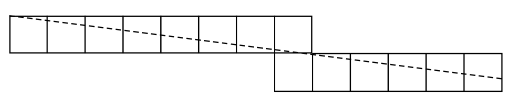
of $2, 3, 5, 7, 11$ or 13. Therefore, to solve this problem one only needs to count the number of factors of $2^3, 3^3, 5^3, 7^3, 11^3, 13^3$ the number 40! contains. The final solution is $(12 + 1) \times (6 + 1) \times (3 + 1) \times (1 + 1) \times (1 + 1) \times (1 + 1) = 2912$.

5. (a) See diagram below. Let $\angle OAB = \alpha$ and $\angle OAE = \beta$. Then each of the angles of the pentagon is $\alpha + \beta$. Since $OA = OB$,$\triangle OAB$ is isosceles. Therefore, $\angle OBA = \alpha$. Moreover, since $\angle ABC = \alpha + \beta$ and $\angle OBA = \alpha$, one has $\angle OBC = \beta$. Similarly, $\angle OCD = \angle ODC = \alpha$ and eventually we get $\angle OEA = \angle OAE = \alpha$. Hence $\alpha = \beta$. Therefore all five triangles in the figure are congruent, and the fives sides of the pentagon are all equal.



(b)

6. We can create mirror images of the initial square billiard board, so that the ball travels in a straight line; as shown below. Then to solve this problem, one counts the number



of copies of the billiard is required for the ball to travel from one corner to another. Since 19 and 96 are co-prime, the number of copies required is $19 + 96 - 1 = 114$.

Senior Questions

1. We use various divisibility rules to limit the number of choose we have. One has

   (a) Divisibility by 5 of the first 5 digits means that the 5th digit has to be 5.

   (b) All even digits must be divisible by either $2, 4, 6$ or 8, hence they all must be even numbers.

   (c) All odd digits must be odd numbers follows from (b).

   (d) Divisibility by 4 of the first 4 digits means that the 3rd and 4th digits combined into a number must be divisible by 4, and using (b), (c) we are limited to the choice $12, 16, 32, 36, 72, 92$ or 96 (last two digits divisible by 4).

   (e) Divisibility by 8 of the first 8 digits means that the 7th and 8th digits combined into a number must be divisible by 4, and using (a), (b) and (c), the $6-7$th digits are limited to $216, 296, 416, 432, 472, 496, 632, 672, 816, 832, 872$ or 896.

   (f) Due to (a), (d) and (e), the $3-8$th digits can only be 125496, 125896, 165432, 165472, 165832, 165872, 325416, 325496, 325816, 325896, 365472, 365872, 725416, 725496, 725816, 725896, 765432, 765832, 925416, 925816, 965432, 965472, 965832 or 965872.

   (g) From (f) and (b), the only choice for 2nd digit are 4 or 8.

   (h) Divisibility by 3 rules on the first three digits implies they must be 147, 183, 189, 381, 387, 741, 783, 789, 981 or 987 (sum of digits divisible by 3).

   Due to $(a)-(h)$ there are only a few numbers to consider, and we can check each of these number by using the divisibility by 7 on the first 7 digits property of the number we are looking for. The only solution is 381654729.

2. By difference of two squares, we have $a^2 - p^2 = (a+p)(a-p)$. There are two cases to consider,

   Case 1: a and p are co-prime. Then a and $a-p$ are co-prime, and a and $a+p$ are co-prime. Similarly, $a+p$ and p^2 are co-prime, $a-p$ and p^2 are co-prime. Therefore, a^2-p^2, p^2 and a are co-prime. Now, since a, b are integers, $b-a$ is an integer, which implies $2ap^2$ must be divisible by a^2-p^2. But since a^2-p^2, p^2 and a are co-prime, 2 is divisible by a^2-p^2. Therefore a^2-p^2 is equal to 1 or 2. There is no solution for this case.

   Case 2: a is divisible by p. Let $a = kp$, where k and p are co-prime. Then

   $$b - kp = \frac{2kp}{k^2-1}.$$

   Since λ and $\lambda \pm 1$ are co-prime, λ and λ^2-1 is co-prime. Hence, $2p$ is divisible by λ^2-1. So that either $\lambda+1$ divides $2p$ or $\lambda-1$ divides $2p$. Since, $\lambda+1 > \lambda-1$, we have three possibilities: $\lambda+1 = p$ and $\lambda-1 = 2$, $\lambda+1 = 2p$ and $\lambda-1 = 1$, or $\lambda+1 = p$ and $\lambda-1 = 1$. The former two possibility yields no solutions. Thus, the only solution is $\lambda = 2$ and $p = 3$. So that $a = 2p = 6$, and $b = \frac{2ap^2}{a^2-p^2} + a = 10$.

3. For any odd integer n,

$$X^n + 1 = (X+1)(X^{n-1} - X^{n-2} + X^{n-3} - \ldots - X + 1). \tag{2}$$

By substituting $X = x^5$ into (2), we see that $x^5 + 1$ is a factor of $x^{15} + 1$. Similar, by substituting $X = x^3$ into (2), we see that $x^3 + 1$ is a factor of $x^{15} + 1$. Moreover, by (2)

$$x^5 + 1 = (x+1)(x^4 - x^3 + x^2 - x + 1) \quad \text{and} \quad x^3 = (x+1)(x^2 - x + 1),$$

so that $x^4 - x^3 + x^2 - x + 1$ and $x^2 - x + 1$ are both factors of $x^{15} + 1$. Therefore, if we can show that $x^4 - x^3 + x^2 - x + 1$ and $x^2 - x + 1$ have no common factors, then their product will form a six degree polynomial that is a factor of $x^{15} + 1$. If the two polynomials have a common factor, then there is a complex x such that

$$x^4 - x^3 + x^2 - x + 1 = 0 \quad \text{and} \quad x^2 - x + 1 = 0.$$

which implies $x^2 - x + 1 = x^4 - x^3 + x^2 = 0 \implies -x + 1 = 0 \implies x = 1$. But this is impossible since $x^2 - x + 1 \neq 0$ when $x = 1$.