**MATHEMATICS ENRICHMENT CLUB.**
**Solution Sheet 9, June 27, 2016**

1. Yes. Let us work backward starting from $A+1$ and applying the two allowed operations multiple times to get to $A$. Put the digit 1 on the left of $A+1$, to obtain a new number $B_1$. Then put the digit 1 on the left of $B_1$ to obtain $B_2$. In this way, recursively putting the digit 1 to the left 8 times, we obtain the number $B_8$. Note that $B_8 - A$ is divisible by 9, because the sum of its digits is divisible by 9. Hence, we can always subtract 9 recursively from $B_8$ to obtain $A$.

2. Consider the triangle with vertices $A = (-a, 0)$, $B = (a, 0)$ and $C = (b, c)$. Let $M$ be the midpoint of $AC$ and $O$ the midpoint of $AB$. Then the median joining $O$ and $C$ is given by
$$y = (b/c)x,$$
and the median joining $M$ and $B$ is given by
$$y = \frac{c}{b-3a}(x-a).$$
Hence, by solving the last two equation simultaneously, we can see that the point of interception of the three medians of the triangle $ABC$ is given by $(b/3, c/3)$; i.e the medians interception of $ABC$ is the average of its coordinates.

   Now by translating the triangle $ABC$ in the first coordinate direction by an amount of $\delta_x$ and in the second coordinate direction by an amount of $\delta_y$. We can see that the resultant triangle will have medians interception at $(b/3 + \delta_x, c/3 + \delta_y)$. In particular, the medians interception of $ABC$ after translation is still the average of its coordinates. Moreover, rotation of triangle about its medians interception fixes its medians interception.

   Since any triangle on a plane can be obtained via translation and rotation, we conclude that the medians interception of any triangle in a plane is given by the average of its coordinates. Therefore, the solution to this problem is $(12/3, 96/3)$.

3. Since $a, b, c$ are the roots of the equation $x^3 + 2x = 1$, we have
$$x^3 + 2x - 1 = (x-a)(x-b)(x-c).$$

Expanding the RHS of the above cubic equation and equating like powers of $x$ gives $a+b+c=0$ and $ab+ab+bc=2$ and $abc=1$. Therefore,

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)=-4.$$

Hence, using the fact that $x^3=1-2x$ for $x=a,b,c$, we have

$$\begin{aligned} a^3+a^2+a+b^3+b^2+b+c^3+c^2+c &= a^2+b^2+c^2-a-b-c+3 \\ &= -4+0+3=-1. \end{aligned}$$

4. There are 4! ways to choose a number leading with one of the digits from $\{1,2,3,4,5\}$. Hence, the total number of possible permutation is $5\times 4! = 120$.

   If we list the 120 possible permutation of numbers, each on top of other, then the number of occurrence of each of the digits $\{1,2,3,4,5\}$ in each column is the same (each unique digit appears $4! = 24$ times). Note that the average value of each column is 3, thus the average of the listed 120 numbers is 3333. Hence their sum is $3333\times 120 = 399960$.

5. The sum of the primes is equal to

$$S = 45 + 9p_{10} + 99p_{100} + \dots, \tag{1}$$

   where $p_{10}$ is the sum of all digits of the primes in the tenth position, and $p_{100}$ is the sum of all digits of the primes in the hundredth position, and so on.

   Since $4,6,8$ are not prime, we must have those digits on the tenth or higher position of the primes. Hence, from (1), it is clear that $p_{10}\geq 4+6+8=18$, and $p_{100}, p_{1000}, \dots$ all zero produces the smallest $S$ possible. In particular, one has $S\geq 45+162=207$. In fact, it is possible to construct a solution such that $S=207$.

6. Since $6! = 6\times 5\times 4\times 3\times 2 = 2^4\times 3^2\times 5$, the maximum value of $n$ will depend on the number of the factors 2, 3 or 5 that is contained in 77!.

   Let $\lfloor x\rfloor$ denote the largest integer smaller than $x$. Since $\lfloor\frac{77}{2}\rfloor=38$, $\lfloor\frac{77}{2^2}\rfloor=19$, $\lfloor\frac{77}{2^3}\rfloor=9$, $\lfloor\frac{77}{2^4}\rfloor=4$, $\lfloor\frac{77}{2^5}\rfloor=2$ and $\lfloor\frac{77}{2^6}\rfloor=1$, we have $77! = 2^{38+19+9+4+2+1}\times A$, where $A$ is a integer containing no factors of 2.

   Similarly, we can find that $77! = 3^{25+8+3+1}\times B$ and $77! = 5^{25+3}\times C$, where $B$ and $C$ are integers, such that $B$, $C$ containing no factors of 3, 5 respectively. Hence, we have $77! = 2^{73}\times 3^{37}\times 5^{28}\times D$, where $D$ is an integer containing no factors of $2,3$ and 5. From this, we conclude that the largest possible $n$ is $\lfloor\frac{73}{4}\rfloor=18$.

**Senior Questions**

1. Here $p$ and $q$ are prime. We have

$$\begin{aligned} p!+1 &= (2p+1)^2 \\ &= 4p^2+4p+1 \\ p! &= 4p^2+4p \\ (p-1)! &= 4p+4 \\ (p-1)!-p+1 &= 3p+5. \end{aligned}$$

Hence

$$\frac{(p-1)!-(p-1)}{p} = \frac{3p+5}{p} = 3+\frac{5}{p}. \qquad (2)$$

Since $p$ is prime, $(p-1)!-(p-1)$ is divisible by $p$. Thus the LHS of (2) is an integer. In particular, $3+5/p$ is an integer, so that $p=5$. The unknown $q$ is found similarly.

2. The substitution $2x = \sec u$ may help.

3. Since $CE$ is perpendicular to $AB$, the triangles $\triangle AEC$ and $\triangle CEB$ are similar. Hence,

   (a) $\angle EAC = \angle ECB$, thus $\angle LAE = \angle MCE$.

   (b) $\frac{CB}{CE} = \frac{AC}{AE}$, thus $\frac{CM}{CE} = \frac{AL}{AE}$.

   It follows that $\triangle LAE$ is similar to $\triangle MCE$. Therefore, $\angle ELA = \angle EMC$, which implies the quadrilateral $LAEM$ is cyclic. Hence, $\angle LEM = \angle LAC = 90°$.