MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 9, July 24, 2017

1. What is the least positive integer $n$ such that $90 \times n$ is a cube?

2. Show that any straight line passing through the centre of a parallelogram (i.e. the intersection of the diagonals) divides the parallelogram into two equal areas.

3. A mathematics test has 5 questions on each of which people can score 0,1,2 or 3 marks. How many ways can a student receive a total of 12 marks for the test?

4. Use the fact that $2xy = (x + y)^2 - x^2 - y^2$ to show that

$$2(b - c)(c - a) + 2(c - a)(a - b) + 2(a - b)(b - c) \leq 0$$

for all real numbers $a, b, c$.

5. Take any triangle $ABC$ and show how to construct an equilateral triangle inside $ABC$ whose vertices touch the sides of $ABC$. (Hint: Start by constructing an equilateral triangle outside $ABC$ with $AB$ as one of its sides.)

6. Imagine that we have a finite set $A$ of integer numbers, that is, a collection of integers without repetition. Consider the set $A + A$ of all possible sums of two numbers in $A$:

$$A + A = \{ n : n = a + a' \text{ for some numbers } a, a' \text{ in } A \}.$$ 

We denote how many numbers there are in the set $A$ by $|A|$.

(a) Show that $|A + A| \geq 2|A| - 1$.

(b) Show that if $|A + A| = 2|A| - 1$, then $A$ is an arithmetic progression.
Senior Questions

1. Imagine that you have a square based cake, like the one in the picture.

   (a) How would you cut it into 5 pieces of equal volume? How about 7 pieces?
   (b) How about \( n \) pieces of equal volume?

2. Show that \( \log_2 3 \) is not a rational number.