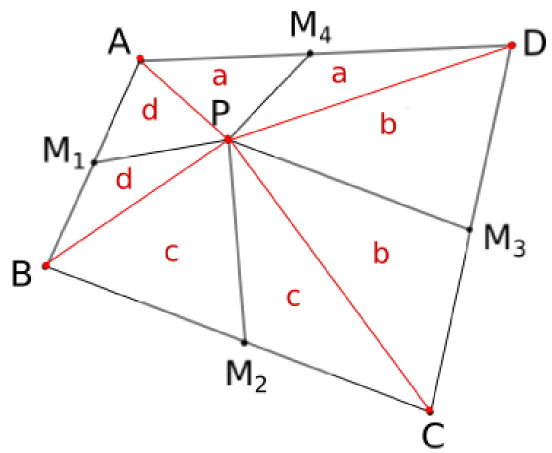
**MATHEMATICS ENRICHMENT CLUB.**
**Solution Sheet 10, July 31, 2017**

1. There are 49 ways, and even more methods of arriving at this answer. Perhaps the easiest is to use cases starting with using $0, 1$ or $2$ possible 50 cent coins.

2. Divide the grid into nine $1 \times 1$ squares. If ten darts are thrown, at least one square contains at least two darts. These darts are less than $\sqrt{2}$ from each other.

3. (a) Both triangles ($ABM$ and $AMC$) have the same height and base size, thus its area is the same.

   (b) Note that if we divide each quadrangle in two triangles by connecting the point $P$ with each of the original corners we can use the previous exercise.




Note that

$$a + d = 7\,\text{cm}^2, \quad a + b = 9\,\text{cm}^2, \quad b + c = 12\,\text{cm}^2 \quad \text{and} \quad d + c = X\,\text{cm}^2.$$

So we have

$$a + b + c + d = (7 + 12)\,\text{cm}^2 = (9 + X)\,\text{cm}^2,$$

so the area we are looking for is precisely: $X = 7 + 12 - 9 = 10\,\text{cm}^2$.

4. (a) $29 = 5^2 + 2^2$, $37 = 6^2 + 1^2$. For 30, note that none of the following are square numbers:

$$301 = 29,\ 304 = 26,\ 309 = 21,\ 3016 = 14,\ 3025 = 5.$$

   Similarly, 31 cannot be expressed as a sum of two squares.

   (b) Easy.

   (c) $1073 = (5^2 + 2^2)(6^2 + 1^2) = (302)^2 + (5 + 12)^2$.
   Swapping $5^2 + 2^2$ with $2^2 + 5^2$ yields $1073 = 7^2 + 32^2$.

5. Pigeon-hole principle. Each number can be written in the form $2^k(2m + 1)$ where $k, m \geq 0$. Since each number is less than 1001, $m$ must be less than 500.

   So since you're choosing 501 numbers, two of the numbers must have the same $m$ value.

   These two numbers can be written as $2^{k_1}(2m + 1)$ and $2^{k_2}(2m + 1)$.

   Either $k_1 \leq k_2$ or $k_2 \leq k_1$, so without loss of generality, assume $k_1 \leq k_2$. Then $2^{k_1}(2m + 1)$ divides $2^{k_2}(2m + 1)$, which concludes the proof.

**Senior Questions**

The number of ways to obtain $k$ when rolling two dices coincides with the coefficient of $x^k$ in

$$f(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^2.$$

Now, note that

$$\begin{aligned} f(x) &= x^{12} + 2x^{11} + 3x^{10} + 4x^9 + 5x^8 + 6x^7 + 5x^6 + 4x^5 + 3x^4 + 2x^3 + x^2 \\ &= (x + x^2)^2(x^3 - x^2 + x)^2(x^3 + x^2 + x)^2. \end{aligned}$$

In order to find two 6-sided dice we need to be able to decompose $f(x) = B(x)C(x)$ as product of two polynomials satisfying:

$$\begin{cases} B(0) = C(0) = 0 & \text{all sides have positive numbers,} \\ B(1) = C(1) = 6 & \text{the dice has exactly 6 sides.} \end{cases}$$

The only possibility is

$$\begin{aligned} B(x) &= (x + x^2)(x^2 + x + 1)(x^2 - x + 1)^2 = x^8 + x^6 + x^5 + x^4 + x^3 + x \\ C(x) &= (x + x^2)(x^2 + x + 1) = x^4 + 2x^3 + 2x^2 + x. \end{aligned}$$

Thus our two dices will have sides $B = 8, 6, 5, 4, 3, 1$ and $C = 4, 3, 3, 2, 2, 1$. And this construction is unique.