MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 12, July 24, 2017

1. The two conditions can be written as

$$\text{(a)} \sum_{d|n} d = \sum_{t|m} t = A, \qquad \text{(b)} \sum_{d|n} \frac{1}{d} = \sum_{t|m} \frac{1}{t} = B,$$

for two positive values A and B (every positive number n has at least two divisors: 1 and n).

Note that if $d|n$ then $\frac{n}{d}$ is also a divisor of n, so in particular

$$nB = n \times \left(\sum_{d|n} \frac{1}{d} \right) = \sum_{d|n} \frac{n}{d} = \sum_{d|n} d = A$$

and similarly for m

$$mB = m \times \left(\sum_{t|m} \frac{1}{t} \right) = \sum_{t|m} t = A.$$

Putting everything together we see that

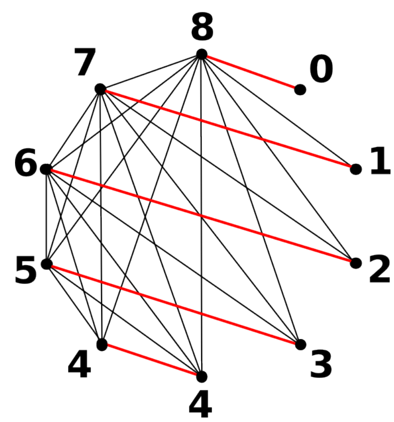
$$n = m = \frac{A}{B}.$$

2. We have to prove that no odd number has a square on it's back and squares have an even number on it's back, so we have to flip cards A and D.

3. There were 10 people in the dinner. Nine of them (all but the matematician) shook hands with different number of people: but, since none of them shook hands with their partners or with themselves, they shook at most 8 hands.

That means that among those 9 guests there has to be one person who shook 8 hands, another shook 7, ..., another shook 1 and someone did not shake hands with any of the guests.

The following diagram express the only possible outcome (black edges represent a handshake, red edges represent the couples and the numbers represent the number of people that each person shook hands with).

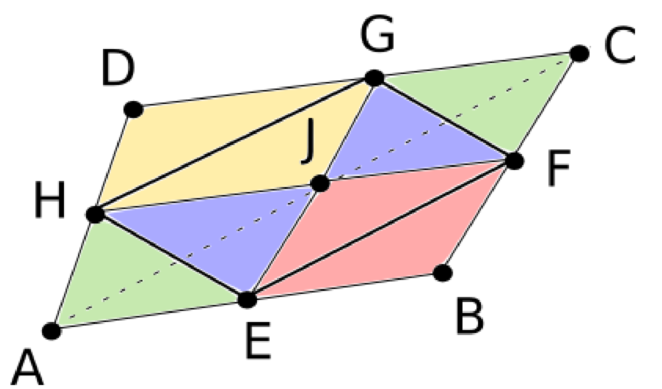


The mathematician and its partner both shook hands with four other people.

4. Let us denote the mid points (like in the picture)

$$E = \frac{A+B}{2}, \quad ,F = \frac{B+C}{2}, \quad ,G = \frac{C+D}{2}, \quad H = \frac{D+A}{2}$$

and in addition let $J = \frac{A+C}{2}$ denote the midpoint of the diagonal AC.



There are two triangles of each color. Each pair of triangles of the same color are congruent to each other, with one of the pair inside parallelogram $EFGH$ and the other outside it. By dissection, $EFGH$ must have exactly half the area of the full quadrilateral $ABCD$.

5. It is easy to see that for any integer value of p either p, $p+2$ or $p+4$ must be divisible by 3, so that means that the triplet $(3, 5, 7)$ is the only possible one.

   To see this, it suffices to consider the cases: $p = 3k+1$ or $p = 3k+2$ (since otherwise p is already divisible by 3).

Senior Questions

1. There are many ways to show this fact. We will use discuss Oresme's proof, that dates back to 1350.

   First note that the partial sums

   $$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

   are monotone increasing, that is:

   $$H_1 = 1 < H_2 < H_3 < \cdots < H_{n-1} < H_{n-1} + \frac{1}{n} = H_N < \cdots < \sum_{k=1}^{\infty} \frac{1}{k}.$$

   It is enough to show that for some particular $m \in \mathbb{N}$ we have $H_m > M$.

   Now, note that

   $$\begin{aligned} H_1 &= 1 = 1 + 0 \times \left(\frac{1}{2}\right) \\ H_2 &= 1 + \frac{1}{2} = 1 + 1 \times \left(\frac{1}{2}\right) \\ H_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 2 \times \left(\frac{1}{2}\right) \\ H_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3 \times \left(\frac{1}{2}\right). \end{aligned}$$

   In fact for any k

   $$H_{2^k} > 1 + k \times \left(\frac{1}{2}\right).$$

   It suffices to take $m = 2^k$ for some $k > 2M - 2$ to conclude that

   $$\sum_{k=1}^{\infty} > H_m > 1 + \frac{k}{2} > 1 + \frac{2M-2}{2} = M.$$

2. Let 2^k be the highest power of 2 smaller or equal than n (so $2^k \leq n$ but $2^{k+1} > n$).

   Imagine that the partial sum was in fact an integer, that is

   $$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \cdots + \frac{1}{n} \in \mathbb{N}.$$

   Note that if $m \leq n$, $m \neq 2^{k-1}$, then

   $$\frac{2^{k-1}}{m}$$

is either an integer (if m was a lower power of two) or it is a rational number with odd denominator. Then clearly

$$2^{k-1}H_n = 2^{k-1} + 2^{k-2} + \frac{2^{k-1}}{3} + 2^{k-3} + \cdots + \frac{1}{2} + \cdots + \frac{2^{k-1}}{n} = \frac{1}{2} + \frac{a}{b}$$

where b is an odd number. Therefore, we can write

$$\frac{1}{2} = \frac{b2^{k-1}H_n - a}{b}$$

but that is impossible provided that b is an odd integer.

We can conclude then that our assumption $S_n \in \mathbb{Z}$ was wrong.

3. Group the terms based on the number of digits in their denominator.

   There are 8 terms in $(1/1 + ... + 1/8)$ each of which is no larger than 1. Consider the next group $(1/10 + ... + 1/88)$.

   The number of terms is at most the number of ways to choose two ordered digits out of the digits 0..8, and each such term is clearly no larger than $1/10$. So this group's sum is no larger than $9^2/10$. Similarly, the sum of the terms in $(1/100 + ... + 1/999)$ is at most $9^3/10^2$, etc.

   So the entire sum is no larger than

$$9 * 1 + 9 * (9/10) + 9 * (9^2/10^2) + ... + 9 * (9^n/10^n) + ...$$

   This a geometric series that sums to 90. Thus by comparison test, original sum (which is smaller term-by-term) must be smaller than 90[1].

[1] There was a mistake in the Problem sheet 12: instead of “Show that in fact the sum of the remainder terms is smaller than 10.” should have said “Show that in fact the sum of the remainder terms is smaller than 100.”