MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 13, August 21, 2017

1. Firstly, we factorise the left hand side of the equation.

$$3x^2 - 8xy + 4y^2 = (3x - 2y)(x - 2y)$$

Let $a = 3x - 2y$, $b = x - 2y$. Then $a - b = 2x$ and $a - 3b = 4y$.
The factors of -12 are $\pm1, \pm2, \pm3, \pm4, \pm6$ and ±12. Using a table to keep track of results, we have

a	b	$a - b = 2x$
1	-12	13 (no solution in $\mathbb{Z}$)
-1	12	-13 (no solution in $\mathbb{Z}$)
2	-6	8
-2	6	-8

and so on. There are four solutions in total: $(4, 5)$, $(-4, -5)$, $(4, 3)$ and $(-4, -3)$.

2. Complete the square, then take difference of two squares. Answers are $(x^2 - 2x + 2)(x^2 + 2x + 2)$ and $(x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$.

3. Suppose $x \le y \le z$. Then $5/8 = 1/x + 1/y + 1/z \le 3/x$, fo $x < 5$. This means there are only 4 possible values for x.
   - $x = 1$: No solution
   - $x = 2$: Solve $1/y + 1/z = 1/8$. So $8 \le y \le 2 * 8$. Testing y values in this range gives $(9, 72)$ and $(10, 40)$
   - $x = 3$: Solve $1/y + 1/z = 7/24$. Since $1/4 < 1/y + 1/z < 1/3$. So $3 \le y \le 2 * 4$. Testing y values in this range gives $(4, 24)$ and $(6, 8)$
   - $x = 4$: Solve $1/y + 1/z = 3/8$. Answers $(3, 24)$ and $(4, 8)$

4. The octagon is not regular!

5. Treat it as an arithmetic progression. Answer is 4.

6. (a) $\tau(7) = 2$, $\tau(10) = 4$ and $\tau(25) = 3$.

(b) If $\tau(m) = 2$, m is a prime; if $\tau(m)$ is odd, then m is a square.

(c) Use the prime factorisation of n. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ for distinct primes, $p_1, \dots, p_k$, then $\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_k + 1)$.

Senior Questions

1. Use partial fractions to express S as a telescoping sum. Thus $\frac{1}{(3n-2)(3n+1)} = \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)$ and so $S = \frac{n}{3n+2}$.

2. (a) Using the hint and integration by substitution with $u = \sin\theta$, You should obtain the result $I = \ln\left(\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}\right) + C$.

   (b) You should obtain the result $I = \ln|\sec\theta + \tan\theta| + C$.

   (c) Two primitives can vary by an arbitrary constant. In this case, the function described by the two primitives is the same (as you can check by graphing them on Wolfram Alpha).

3. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ for distinct primes, $p_1, \dots, p_k$, then

$$\sigma(n) = (1 + p_1^1 + p_1^2 + \dots + p_1^{\alpha_1})(1 + p_2^1 + p_2^2 + \dots + p_2^{\alpha_2}) \dots (1 + p_k^1 + p_k^2 + \dots + p_k^{\alpha_k})$$