MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 1, May 7, 2018

1. How many integers between 100 and 999 have distinct odd digits?

2. The number $23AB3$ is exactly divisible by 99. What are the numbers $A$ and $B$?

3. Let $K$ be the circumcircle through the vertices of a rectangle with sides $a$ and $b$. On each side of the rectangle construct a semicircle. This will give four crescents formed between these semicircles and $K$. What is the sum of the areas of the four crescents?

4. Suppose the last digit of $x^2 + xy + y^2$ is zero, and $x$ and $y$ are positive integers. Prove that the last two digits of $x^2 + xy + y^2$ are both zero.

5. Given a triangle $ABC$, draw a straight line through each vertex parallel to the opposite side, thereby forming a new triangle $A'B'C'$ with $A'$ opposite $A$ and so on, as shown in the diagram below.

\[\begin{array}{c}
\text{B'} \\
\text{A} \\
\text{C'} \\
\hline
\text{C} \\
\text{B} \\
\text{A'}
\end{array}\]

(a) Show that the altitude drawn from $A$ in the triangle $ABC$ is the perpendicular bisector of $B'C'$. (Hint: Look for parallelograms.)

(b) Conclude that the three altitudes of a triangle are concurrent.
Senior Questions

1. Let \( I = \int_{0}^{1} \frac{x^4(x - 1)^4}{x^2 + 1} \, dx \). By evaluating \( I \), deduce that \( \pi < \frac{22}{7} \).

2. A continuous function \( f \), maps the interval \([0, 1]\) into \([0, 1]\). Show that there is a real number \( \alpha \) in this interval such that \( f(\alpha) = \alpha \). (A diagram is not sufficient.)

3. The function \( f(x) = x^x \) has an inverse \( g(x) \) provided we restrict the domain of \( f \) to \( x > 1 \). Find a formula for the derivative of \( g(x) \) in terms of \( x \) and \( g(x) \).

4. Let \( P(x) \) be a polynomial with integer coefficients that satisfies \( P(17) = 10 \) and \( P(24) = 17 \). Given that \( P(n) = n + 3 \) has two distinct integer solutions \( n_1 \) and \( n_2 \), find the product \( n_1 \cdot n_2 \). [2005 AIME II Problem 13]