MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 1, May 7, 2018

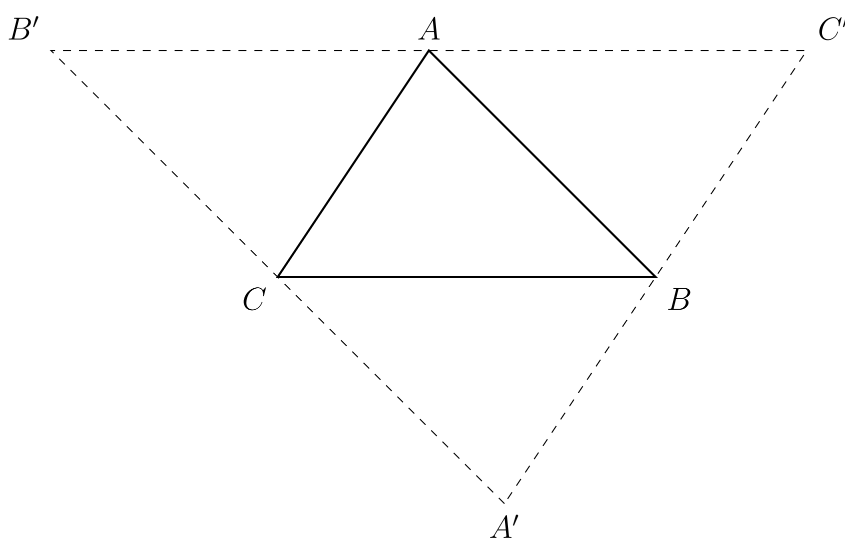
1. How many integers between 100 and 999 have distinct odd digits?

2. The number $23AB3$ is exactly divisible by 99. What are the numbers A and B?

3. Let K be the circumcircle through the vertices of a rectangle with sides a and b. On each side of the rectangle construct a semicircle. This will give four crescents formed between these semicircles and K. What is the sum of the areas of the four crescents?

4. Suppose the last digit of $x^2 + xy + y^2$ is zero, and x and y are positive integers. Prove that the last **two** digits of $x^2 + xy + y^2$ are both zero.

5. Given a triangle ABC, draw a straight line through each vertex parallel to the opposite side, thereby forming a new triangle $A'B'C'$ with A' opposite A and so on, as shown in the diagram below.



   (a) Show that the altitude drawn from A in the triangle ABC is the perpendicular bisector of $B'C'$. (Hint: Look for parallelograms.)

   (b) Conclude that the three altitudes of a triangle are concurrent.

Senior Questions

1. Let $I = \int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,dx$. By evaluating I, deduce that $\pi < \frac{22}{7}$.

2. A continuous function f, maps the interval $[0,1]$ into $[0,1]$. Show that there is a real number α in this interval such that $f(\alpha) = \alpha$. (A diagram is not sufficient.)

3. The function $f(x) = x^x$ has an inverse $g(x)$ provided we restrict the domain of f to $x > 1$. Find a formula for the derivative of $g(x)$ in terms of x and $g(x)$.

4. Let $P(x)$ be a polynomial with integer coefficients that satisfies $P(17) = 10$ and $P(24) = 17$. Given that $P(n) = n+3$ has two distinct integer solutions n_1 and n_2, find the product $n_1 \cdot n_2$. [2005 AIME II Problem 13]