



MATHEMATICS ENRICHMENT CLUB.

Problem Sheet 10, July 30, 2018

1. A sports club has a total of 163 members. The club offers a choice of basketball, cricket and soccer (or a combination of these three). Each member selects one or more activity to do. Of the club members, 73 play cricket, 100 play basketball, 60 play at least two different sports and 10 play all three.
 - (a) How many members play soccer?
 - (b) There are 25 members that play both basketball and cricket, how many members play soccer *only*?
2.
 - (a) Show that if a whole number is divisible by 4, then so is the number formed by its last two digits.
 - (b) Show that if a whole number is divisible by 9, then so is the sum of its digits.
3. Show that, for any whole number $n > 1$,

$$\frac{n-1}{n} - \frac{n-2}{n-1} = \frac{1}{n(n-1)}.$$

Hence calculate

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{10100}.$$

4. A continued fraction has the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}},$$

where a_0 is a non-negative integer and a_1, a_2, a_3, \dots are positive integers. It is generally easier to write a continued fraction in the form $[a_0; a_1, a_2, a_3, \dots]$. A continued fraction may either terminate or be an infinite expression.

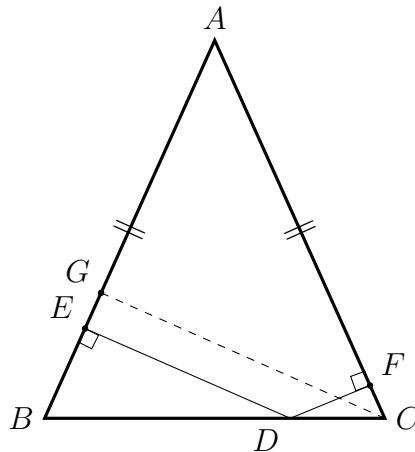
Question 4 continues over the page.

Find the continued fraction expression for the following numbers. What do you notice?

- (a) $\frac{355}{113}$
- (b) $\frac{113}{355}$
- (c) $\sqrt{2}$
- (d) $\frac{1}{\sqrt{2}}$

Hint: a_0 is the integer part of the number.

5. (a) Let ABC be an isosceles triangle with BC the base. Let D be an arbitrary point on BC . Let E and F be the feet of the perpendiculars drawn from D to the lateral sides AB and AC , respectively. Let CG be the altitude from C to AB . Show that $ED + DF = CG$.



- (b) How does this theorem change if D lies on the extension of the base BC ?
- (c) Prove that, in an equilateral triangle, the sum of the distances from any interior point to the sides of this triangle does not depend on the point, and is congruent to the altitude of the triangle.¹

¹Adapted from AP Kiselev *Kiselev's Geometry: Planimetry*, Tr. A Givental, 2006.

Senior Questions

1. A polynomial (of degree n) is a function that can be written in the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

where $a_n \neq 0$. That is, a polynomial is the finite sum of monomial terms of the form $a_k x^k$ in which the variable x is raised to a non-negative integer power. The largest value of k for which $a_k \neq 0$ is called the degree of the polynomial.

Let p and q be non-zero polynomials.²

- (a) Show that $\deg(p + q) \leq \max(\deg p, \deg q)$. When would strict inequality hold?
- (b) What is the relationship between $\deg(p \circ q)$ and $\deg(q \circ p)$? Note: $p \circ q(x) = p(q(x))$.
- (c) Show that, for $x > 0$, the function $f(x) = \log x$ is not a polynomial.
Hint: consider $\log(x^2)$.

²This question is adapted from EJ Barbeau, *Polynomials*, Springer, 1989.