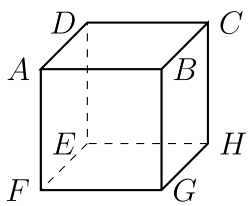
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 13, August 20, 2018

1. In the equation
$$29 + 38 + 10 + 4 + 5 + 6 + 7 = 99,$$
the left hand side contains each digit exactly once. Either find a similar expression using all the digits from one to nine and only addition signs to obtain 100 or prove that it isn't possible.

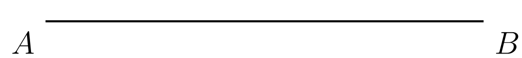
2. Let $ABCDEFGH$ be a cube.



   (a) A triangle can be created by joining any three distinct vertices of the cube. How many such triangles are there?
   (b) How many, if any, of these triangles are acute?

3. A bakery sells donuts in packs of 5, 9 or 13. What is the largest number of donuts that cannot be bought exactly?

4. **Construction problem:** Suppose that we are given a line segment $AB = d + s$ where d is the length of the diagonal and s is the length of the side of the square.



   Explain how to construct the square using compass and straight-edge techniques.

   Note: you don't need to actually do the constructions, you just need to explain *how* to do them, and, just as importantly, prove that your construction works.[1]

5. Let $S = \{1, 2, 3, \ldots, 17\}$. Let T be a subset of S that has exactly 8 elements. From each such T, select the smallest number, x_{min}. Show that $\bar{x}$, the arithmetic mean of these 24 310 smallest elements, is 2.

[1] Adapted from AP Kiselev *Kiselev's Geometry: Planimetry*, Tr. A Givental, 2006

Senior Questions

1. Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx,$$

where a, b, and c are constants.

   (a) Show that the x coordinates of the stationary points of the cubic satisfy the equation

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}.$$

   (b) Hence find conditions on a, b, and c such that the cubic has no stationary points. If $b = 0$, under what conditions will the cubic have stationary points?

   (c) Show that, when the cubic has stationary points, the x-coordinate of the point of inflexion is the average of the x coordinates of the two stationary points.

2. Let $x = \sqrt[3]{10} + \sqrt[3]{6}$. Show that $x^3 - 3x\sqrt[3]{60} = 16$ and deduce (without a calculator) that $x < 4$.