**MATHEMATICS ENRICHMENT CLUB.**
**Problem Sheet 6, June 11, 2018**

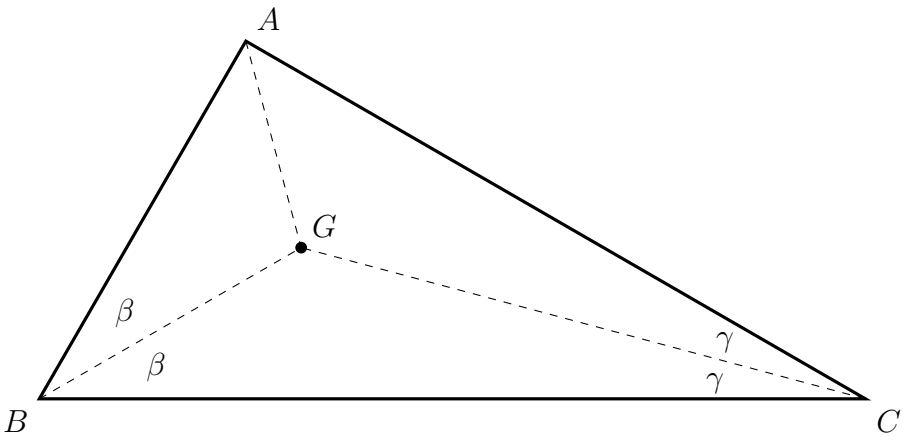
1. Solve $\dfrac{x+3y}{2x+5y} = \dfrac{4}{7}$.

2. Let $n$ be a positive integer such that $100n + 64$ and $201n + 64$ are both 4-digit square numbers. Find $n$.

3. Is it possible to cut a square into nine squares and colour one of them white, three of them grey and five of them black, such that squares of the same colour have the same size and squares of different colours will have different sizes?

4. (a) What are the possible remainders when we divide a square integer by 4?
   (b) Suppose we have three numbers such that the sum of any two of them is a square. Show that at most one of the numbers can be odd.
   (c) Can you find such an example with exactly one of the numbers being odd?

5. Let $ABC$ be a triangle. Let $G$ be the point of intersection of angle bisectors $BG$ and $BC$. Prove that $AG$ also bisects $\angle BAC$. (The point $G$ is called the *incentre* of the triangle $ABC$.)



**Senior Questions**

1. Prove that the square of the $n$th triangular number is the sum of the first $n$ cubes, i.e.

$$\left(\sum_{k=1}^{n} k\right)^2 = \sum_{k=1}^{n} k^3.$$

2. Find the limit $\lim_{n\to\infty} \dfrac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}$.

3. Let $f(x) = xe^x$.

   (a) Draw the graph of $y = f(x)$, clearly indicating any stationary points on your diagram.

   (b) For $x \geq -e^{-1}$, $f(x)$ has an inverse, $f^{-1}(x)$. Add the graph of $y = f^{-1}(x)$ to your diagram.

   (c) This inverse is the principal branch of the Lambert W function, and is also known as the Omega function or the product-log function. We will denote it by $W(x)$. Show that

$$\frac{dW}{dx} = \frac{W(x)}{x(1 + W(x))}.$$