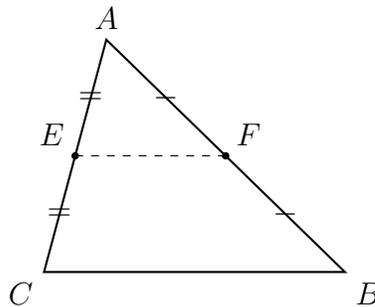


**MATHEMATICS ENRICHMENT CLUB.**

**Problem Sheet 8, June 25, 2018**

- Without using a calculator, show that the number 13950264876 is not a square by considering divisibility by three.
- Find the largest positive integer that when divided into each of 364, 414 and 539 leave the same remainder.
- The mid-line theorem.** Suppose that  $ABC$  is a triangle with  $E$  and  $F$  being the midpoints of  $AC$  and  $AB$ , respectively. Prove that  $EF$  is parallel to  $BC$  and half the length of  $BC$ .



- How many acute angles can a convex polygon have?
- Find infinitely many integers  $x$  such that
 
$$\sqrt[2]{x + \sqrt{x^2 + 1}} + \sqrt[3]{x - \sqrt{x^2 + 1}}$$
 is an integer.
- The *phi* function of a positive integer  $n$ ,  $\phi(n)$ , returns the number of positive integers less than and coprime to  $n$  (that is, integers which have no common factor with  $n$  except 1). For example, there are four positive integers less than 5 which are coprime with it (1, 2, 3 and 4), so  $\phi(5) = 4$ . In contrast, there are only two integers less than 6 that are coprime with it (1 and 5), so  $\phi(6) = 2$ .
  - Find  $\phi(12)$  and  $\phi(30)$ .
  - Suppose that  $p$  is prime. Find  $\phi(p)$ ,  $\phi(p^2)$  and  $\phi(p^3)$ .
  - If  $p$  and  $q$  are two distinct primes, find  $\phi(pq)$ .

### Senior Questions

- Show that  $n^4 - 6n^3 - 18n^2 + 6n + 1 = (n^2 - 3n - 1)^2 - 25n^2$ ,
  - Hence find all integers  $n$  such that  $n^4 - 6n^3 - 18n^2 + 6n + 1$  is prime.
- How many real roots does the equation  $x = 3\pi(1 - \sin x)$  have? Use Newton's method to find an approximate value of the smallest one and hence find the largest one.
- Let  $ABC$  be a triangle. The median of a triangle is the line segment that connects the midpoint of one side to the opposite vertex. Prove that the medians of  $ABC$  intersect at a single point, called the centroid, and that the centroid divides the median in the ratio  $1 : 2$ , with the centroid lying twice as far from the vertex as from the foot of the median.