**MATHEMATICS ENRICHMENT CLUB.**
**Solution Sheet 1, May 7, 2018**

1. There are 5 odd integer digits to choose from. Once one is chosen for the first digit, only 4 remain, then 3. So there are $5 \times 4 \times 3 = 60$ 3-digit numbers with distinct odd digits.

2. Use the divisibility rules for 9 and 11 to set up a system of simultaneous equations. $A = 4$ and $B = 6$.

3. The radius of the circumcircle is $\sqrt{a^2 + b^2}$. The sum of the area of the 4 crescents, is the sum of the 4 semi circles, plus the area of the rectangle, then with the area of the circumcircle subtracted. So

$$\begin{aligned} A &= 2 \cdot \frac{\pi}{2}\left(\frac{a}{2}\right)^2 + 2 \cdot \frac{\pi}{2}\left(\frac{b}{2}\right)^2 + ab - \pi\left(\frac{\sqrt{a^2+b^2}}{2}\right)^2 \\ &= ab + \pi\left(\frac{a^2}{4} + \frac{b^2}{4} - \frac{a^2}{4} - \frac{b^2}{4}\right) \\ &= ab \end{aligned}$$

4. If either $x$ or $y$ is odd, $x^2 + xy + y^2$ is also odd. Hence they are both even. If one is a multiple of 10 and the other is not, $x^2 + xy + y^2$ is not a multiple of 10. Suppose both $x$ and $y$ are not multiples of 10. Then $x^2$ and $y^2$ end in 4 or 6, while $xy$ cannot end in 0. So we cannot have one of $x^2$ or $y^2$ ending in 4 and the other in 6. If $x^2$ and $y^2$ both end in 4 or both end in 6, then $xy$ must also end in 4 or 6 and so $x^2 + xy + y^2$ is not a multiple of 10. So the only possibility is that $x$ and $y$ are both multiples of 10, meaning $x^2 + xy + y^2$ is a multiple of 100.

5. (a) $ABCB'$ is a parallelogram since $BC$ is parallel to $AB'$ and $CB'$ is parallel to $AB$. Similarly $CBC'A$ is a parallelogram. So now we know that $A$ is the midpoint of $B'C'$. Now $\angle B'AC = \angle ACB$ because they are alternate. If $D$ is the point at which the altitude from $A$ meets $BC$ then $\angle DAC = 90^\circ - \angle ACD = 90^\circ - \angle ACB$ so $\angle DAC + \angle B'AC = 90^\circ$, and $AD$ is the perpendicular bisector of $B'C'$.

   (b) Since all the altitudes are also perpendicular bisectors of a triangle, and perpendicular bisectors of a triangle are concurrent, these altitudes are also.

**Senior Questions**

1. Use polynomial long division to show that

$$\frac{x^4(x-1)^4}{x^2+1} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}.$$

Use this result to show that $I = \frac{22}{7} - \pi$. Since the integrand is positive on $[0,1]$, so is the integral $I$ and the result follows.

2. If $y = g(x)$, then

$$\begin{aligned} x &= y^y \\ &= e^{\ln(y^y)} \\ &= e^{y\ln(y)} \end{aligned}$$

By the chain rule,

$$\begin{aligned} \frac{dx}{dy} &= (\ln(y)+1)e^{yln(y)} \\ &= (\ln(g(x))+1)x \end{aligned}$$

If $x > 1$, this last expression is not zero, so we can take reciprocals to obtain

$$\frac{dy}{dx} = \frac{1}{(\ln(g(x))+1)x}.$$

3. Consider the function $g(x) = f(x) - x$. Then $g(0) = f(0)$. If $f(0) = 0$, then $\alpha = 0$. So suppose that $f(0) \neq 0$. Since $f$ maps $[0,1]$ to $[0,1]$, this means that $f(0) > 0$. Similarly, consider $g(1) = f(1) - 1$. If $f(1) = 1$ then $\alpha = 1$, but if $f(1) \neq 1$, then $g(1) < 0$. Since $f$ is a continuous function, then so is $g$, and since $g(0)$ and $g(1)$ have opposite signs, $g$ has a zero in $[0,1]$. That is, there is a number $\alpha \in [0,1]$ such that $f(\alpha) = \alpha$.

4. Let $Q(x) = P(x) - x + 7$, noting that $Q(x)$ has roots 17 and 24. Hence

$$P(x) - x + 7 = A(x-17)(x-24).$$

In particular, this means that

$$P(x) - x - 3 = A(x-17)(x-24) - 10.$$

Therefore, $x = n_1, n_2$ satisfy $A(x-17)(x-24) = 10$, where $A$, $(x-17)$, and $(x-24)$ are integers. This cannot occur if $x \leq 17$ or $x \geq 24$ because the product $(x-17)(x-24)$ will either be too large or not be a divisor of 10. Thus $x = 19$ and $x = 22$ are the only values that allow $(x-17)(x-24)$ to be a factor of 10. Hence the answer is $19 \cdot 22 = 418$.