MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 10, 6 August, 2018

1. Let S be the number of members that play Soccer.

   (a) If we add the number of members that play either Basketball, Cricket or Soccer, we end up with a number that is greater than the total number of members in the sports club, because we have double counted the number of members that plays two sports *only*, and triple counted the number of members that plays *all* three.

   So to balance this out we need to subtract the double/triple counts: We know that 10 members play *all* three sports, so these members we triple counted. There are 60 members that plays two or more sports, and 10 that plays all three, therefore there are $60 - 10 = 50$ members that plays two sports *only*.

   The balanced equation is then

   $$163 = S + 100 + 73 - 50 - 2(10),$$

   which gives $S = 60$.

   (b) The number of members that play both Basketball and Cricket but not Soccer is $25 - 10 = 15$, therefore $60 - 15 = 45$ members plays Soccer and Basketball or Soccer and Cricket or all three sports. Since $S = 60$, $60 - 45 = 15$ of these members play Soccer only.

2. (a) Let $a_1, a_2, \ldots a_k$ be the digits of a k digit long whole number n. Then

   $$n = 10^k a_k + 10^{k-1} a_{k-1} + \ldots + 10^2 a_3 + 10a_2 + a_1.$$

   Since 10^i is divisible by 4 for $i = 2, 3, \ldots k$, if n is divisible by 4, then so is $10a_2 + a_1$, which is the number formed by the last two digits of n.

   (b) Let m be the number formed by the sum of all of the digits of n; that is

   $$m = a_k + a_{k-1} + \ldots + a_2 + a_1.$$

   Consider the difference

   $$n - m = (10^k - 1)a_k + (10^{k-1})a_{k-1} + \ldots + 99a_3 + 9a_2.$$

   Clearly $n - m$ is a multiple of 9, so if n is divisible by 9, then so is m.

3. We can write the finite sum as

$$1+\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{10100}=1+\sum_{n=2}^{101}\frac{1}{n(n-1)}$$

Using the given formula,

$$\begin{aligned}1+\sum_{n=2}^{101}\frac{1}{n(n-1)} &= 1+\sum_{n=2}^{101}\left(\frac{n-1}{n}-\frac{n-2}{n-1}\right)\\ &= 1+\sum_{n=2}^{101}\frac{n-1}{n}-\sum_{n=1}^{100}\frac{n-1}{n}\\ &= 1+\frac{100}{101}.\end{aligned}$$

4. Let the number we wish to express as a continued fraction be n. As given in the hint, $a_0=\lfloor n\rfloor$, which is easy to calculate. Then

$$n-a_0=\cfrac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3+\ldots}}},$$

and taking reciprocals, we have

$$\frac{1}{n-a_0}=a_1+\cfrac{1}{1+\frac{1}{a_2+\frac{1}{a_3+\ldots}}}.$$

Once again, we can see that a_1 is the integer part of $\dfrac{1}{n-a_0}$. By iteratively taking reciprocals and finding integer parts, we can determine the values of $a_1,a_2,a_3,\ldots$. You should obtain the following results:

(a) $\dfrac{355}{113}=[3;7,16]$

(b) $\dfrac{113}{355}=[0;3,7,16]$

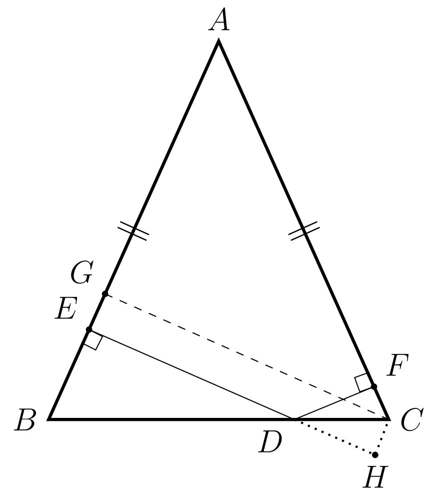
(c) $\sqrt{2}=[1;2,2,2,\ldots]$

(d) $\dfrac{1}{\sqrt{2}}=[0;1,2,2,2,\ldots]$

You should notice that (a) and (b) have terminating continued fractions, whereas (c) and (d) have repeating infinite continued fractions. This is, in fact, generally true: rational numbers have terminating continued fractions and quadratic irrationals have infinite continued fractions that repeat.

Also note that (a) and (b) are reciprocals (as are (c) and (d)), and their continued fractions are closely related.

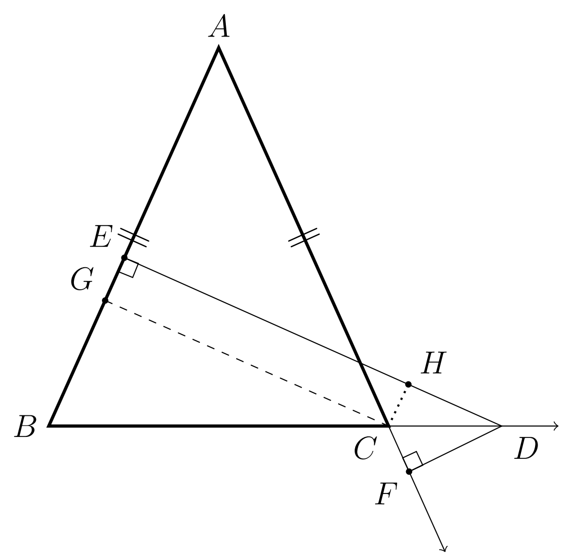
5. (a) Let H be the reflection of F in the line BC. Then $DH=DF$; $\angle FDC=\angle CDH$; and $\angle DFC=\angle DHC=90^\circ$.



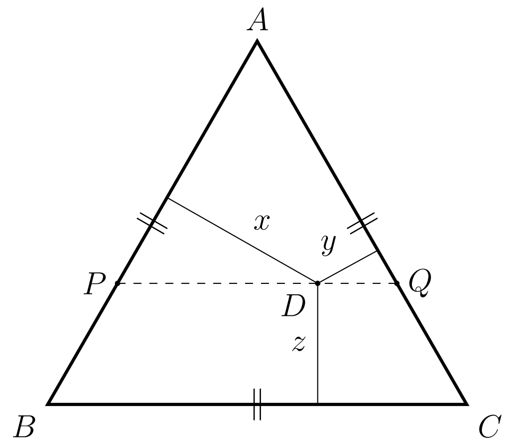
Now $\angle ABC = \angle ACB$, since $\triangle ABC$ is isosceles, and since $\triangle EBD$ and $\triangle DFC$ are both right triangles, $\angle EDB = \angle FDC = \angle CDH$. Thus $\angle EDB$ and $\angle CDH$ are vertically opposite and hence EH is a straight line.

Furthermore, $\angle EGC$, $\angle GEH$ and $\angle DHC$ are all right angles, so $EGCH$ is a rectangle. Hence $CG = EH + DH = ED + DF$.

(b) The altitude is equal to the difference of the two distances.



(c) Let ABC be an equilateral triangle. Let D be an arbitrary point inside ABC, with distances to the vertices x, y and z, as shown.



Let PQ be a line parallel to BC that passes through D, as shown. Thus the distance between BC and PQ is z. Note that $\triangle APQ$ is equilateral, and hence also isosceles. Thus we can use the result of part (a) to show that $x+y$ equals the length of the altitude from AP to Q. Since $\triangle APQ$ is equilateral, all the altitudes are of equal length, and thus the altitude from A to PQ also has length $x+y$. Furthermore, as the altitude of $\triangle ABC$ is equal to the perpendicular distance from A to BC, this is equal to the sum of the altitude of $\triangle APQ$ and z, and hence the sum of x, y and z.

Senior Questions

1. (a) Suppose that p has degree n and q has degree m. Without loss of generality, we may suppose that $n \geq m$. Then

$$\begin{aligned} p(x) &= p_n x^n + p_{n-1}x^{n-1} + \ldots + p_0 \\ q(x) &= q_n x^n + q_{n-1}x^{n-1} + \ldots + q_0, \end{aligned}$$

where $q_k = 0$ for any term such that $k > m$. Then

$$(p+q)(x) = (p_n + q_n)x^n + (p_{n-1} + q_{n-1})x^{n-1} + \ldots + (p_0 + q_0)$$

Now if $p_n + q_n \neq 0$, $\deg(p+q) = n = \max(\deg p, \deg q)$. If $p_n + q_n = 0$, then $\deg(p+q) < n = \max(\deg p, \deg q)$. So we can see that there will be strict inequality if $n = m$ and the leading coefficients of p and q cancel.

(b) $\deg(p \circ q) = \deg(q \circ p) = (\deg p)(\deg q)$.

(c) Suppose that $\log x$ is a polynomial of degree n. Now consider $\log(x^2)$. Since x^2 is a quadratic, it has degree 2, so by part (b), $\log(x^2)$ is a polynomial of degree $2n$. By the log laws, we know that

$$\log(x^2) = 2\log(x),$$

but the right hand side has degree of only n. Thus $2n = n$, which would imply that $n = 0$. But this would mean that the log function is a constant, which is false. Therefore, $\log(x)$ is not a polynomial.