1. Let $BE$ and $CF$ be perpendiculars dropped from $B$ and $C$ to $AM$, extended if necessary. We need to prove that $BE = CF$.

Since $BE$ and $CF$ are both perpendicular to $AM$, $\angle BED = \angle DFC = 90^\circ$, and since $AM$ is a median, $BM = CM$. Moreover, $\angle BDE = \angle CDF$, since $\angle BDE$ and $\angle CDF$ are vertically opposite. Thus $\triangle BDE \equiv \triangle CDF$ by AAS. Thus $BE$ and $CF$ are corresponding sides in congruent triangles and hence equal.

2. (a) To begin, $n^5 - 5n^3 + 4n$ can be factored as $(n + 2)(n + 1)n(n - 1)(n - 2)$. That is, if $n$ is an integer, then $n^5 - 5n^3 + 4n$ is the product of five consecutive integers, and hence can be divided by each of 5, 4, 3 and 2. Thus it must be divisible by 120.

(b) Conversely, suppose that 49 is a divisor of $n^2 + n + 2$ for some integer $n$. Then $n^2 + n + 2 = (n + 4)^2 - 7(n + 1)$ and both $(n + 4)^2$ and $7(n + 2)$ must be divisible by 49, or both $(n + 4)$ and $(n + 2)$ by 7. This is not possible.

3. If A was truthful about B coming second, then B must be lying about A coming second and C about B coming third, so the order would be ABC.

If A was truthful about C coming first, then B must be lying that C was third and C about A coming first, so the order would be CAB. Either way, A beat B.
4. (a) $0.75_{10} = 0.11_2$, since $0.75 = \frac{1}{2} + \frac{1}{4} = 1 \times \frac{1}{2^1} + 1 \times \frac{1}{2^2}$

(b) $0.96875_{10} = 0.11111_2$ in base 2.

(c) 
\[
\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \cdots = 0.\dot{1}_2 = 1
\]

If you are not convinced of this last fact, let $x = 0.\dot{1}_2$. Then

\[
\begin{align*}
2x &= 1.\dot{1}_2 \\
x &= 0.\dot{1}_2 \\
\hline
x &= 1
\end{align*}
\]

(1) \hspace{1cm} (2)

5. Firstly, we note that $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. Then, we find the prime factorisation of $1729 = 7 \times 13 \times 19$. Thus the possible factors of 1729 are 1, 7, 13, 19, 91, 133, 247, and 1729 itself. If we assume that $x - y = 1$, then

\[x^2 + xy + y^2 = 1729.\]

Furthermore, we can substitute $x = y + 1$ into this second equation, thereby obtaining a quadratic in $y$. In this case, the quadratic does not have integer solutions, as $\Delta$ is not a perfect square. However, continuing this way through all the possibilities, we obtain the solutions $(-1, 12), (1, -12), (-9, 10)$ and $(9, -10)$.

Senior Questions

1. (a) i. $(1, 1)$ ii. $(0, -1)$ iii. $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$ iv. $(-\sqrt{3}, 1)$

(b) The graphs should be as follows:

i. $r = \theta$ ii. $r = \cos(2\theta)$ iii. $r = \sin(3\theta)$ iv. $r = 1 + 2\cos \theta$
2. Join \( CP \) and \( PB \) as shown.

Let \( \angle ACB = \alpha \), \( \angle BCP = \beta \) and \( \angle FDP = \gamma \). Let \( D \) and \( E \) be the feet of perpendiculars from \( P \) to the sides of the triangle as shown. Extend a line through \( D \) and \( E \), and let \( F \) be the point of intersection of \( DE \) with the side \( AB \) (extended if necessary). We have to show that \( \angle BFP = 90^\circ \).

Now \( \angle PDC = \angle PEC = 90^\circ \), so \( DECP \) is a cyclic quadrilateral. Thus \( \angle DEP = \angle DCP = \beta \). Furthermore, by the angle sum of \( \triangle ECG \), \( \angle EGC = 90^\circ - \alpha \). But \( \angle EGC = \angle GDE + \angle DEG \), by the exterior angle theorem. And this implies that \( \angle EDG = 90^\circ - (\alpha + \beta) \). Since \( \angle BDF \) and \( \angle EDG \) are vertically opposite, \( \angle BDF = \angle EDG = 90^\circ - (\alpha + \beta) \). Consequently, \( \gamma = \alpha + \beta \).

Since \( BACP \) is a cyclic quadrilateral \( \angle ABC + \angle ACP = 180^\circ \). Thus \( \angle ABP = 180^\circ - \angle ACP = 180^\circ - (\alpha + \beta) \), which implies that \( \angle FBP = \alpha + \beta = \gamma \). Thus \( \angle FBP = \angle FDP \), and so \( FBDP \) is a cyclic quadrilateral also. Hence \( \angle BFP + \angle BDP = 180^\circ \), and so \( \angle BFP = 90^\circ \), as required.