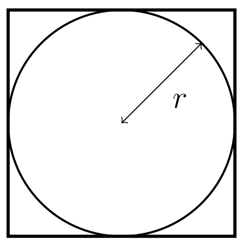
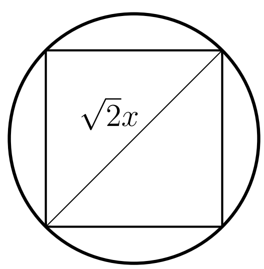
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 12, 20 August, 2018

1. Suppose that the round pizza has radius r, and fits perfectly into a square box.



The box's sides must then be $2r$, so the ratio of pizza area to box area is $\frac{\pi r^2}{4r^2} = \frac{\pi}{4} \approx 0.79$. If the square pizza with side length x fits neatly into a circular box, the diameter of the box would be $\sqrt{2}x$.



Thus the ratio of pizza area to box area is $\frac{x^2}{x^2\pi/2} = \frac{2}{\pi} \approx 0.64$. Thus the more economical use of space (where the pizza fills more of the box) is the first option.

2. Suppose the two numbers were $a < b$ and the incorrect result c. Then $ab - 70 = c$ and $\frac{c}{a} = 48 + \frac{17}{a}$. So $c = 48a + 17$, which means $ab = 48a + 17 + 70$. Thus $a(b - 48) = 87$. The only two factors of 87, other than 1 and 87 itself, are 3 and 29. Now, both a and b are two digit numbers, and so $a = 29$ and $b - 48 = 3$, which means that $b = 51$.

3. Firstly, note that x, y and z are non-zero. Rearranging the equation gives

$$\begin{aligned}\frac{1}{x} + \frac{1}{y} &= \frac{1}{x+y+z} - \frac{1}{z}\\ \frac{x+y}{xy} &= -\frac{(x+y)}{z(x+y+z)}\end{aligned}$$

So we have to satisfy

$$z(x+y)(x+y+z) = -xy(x+y).$$

Consequently,

$$\begin{aligned} z(x+y)(x+y+z) + xy(x+y) &= 0 \\ (x+y)[z(x+y+z) + xy] &= 0 \\ (x+y)(zx + zy + z^2 + xy) &= 0 \end{aligned}$$

We can factorise the second bracket using grouping in pairs to obtain

$$(z+y)(y+z)(x+z) = 0$$

So as long as two of x, y and z are equal in magnitude but opposite in sign, then the remaining pronumeral can take any non-zero integer value at all.

4. Let's write $N = 100a + 10b + c$ or $[abc]$ for short. The

five numbers that can be obtained by permuting the digits are $[acb]$, $[bac]$, $[bca]$, $[cab]$ and $[cba]$. We know that

$$\frac{1}{5}([acb] + [bac] + [bca] + [cab] + [cba]) = N.$$

Since adding the mean to a set of numbers doesn't change the mean, so we can write

$$\frac{1}{6}([abc] + [acb] + [bac] + [bca] + [cab] + [cba]) = N.$$

Adding up the left hand sides, we have

$$2(a+b+c) \times 100 + 2(a+b+c) \times 10 + 2(a+b+c) \times 1 = 6N.$$

Thus

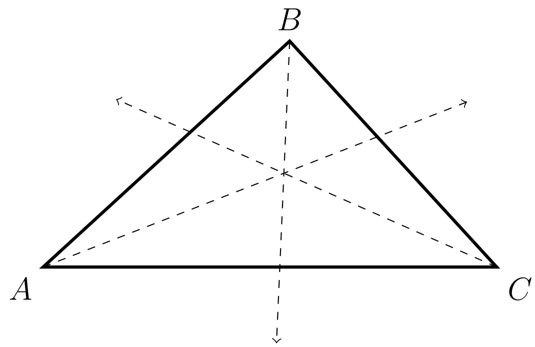
$$111(a+b+c) = 3N.$$

Dividing this by 3, we have $N = 37(a+b+c)$. Since $N < 500$, $a+b+c \leq 13$. Also, a, b and c must be distinct and non-zero, so $a+b+c \geq 1+2+3 = 6$. The multiples of 37 in this range that have distinct non-zero digits are:

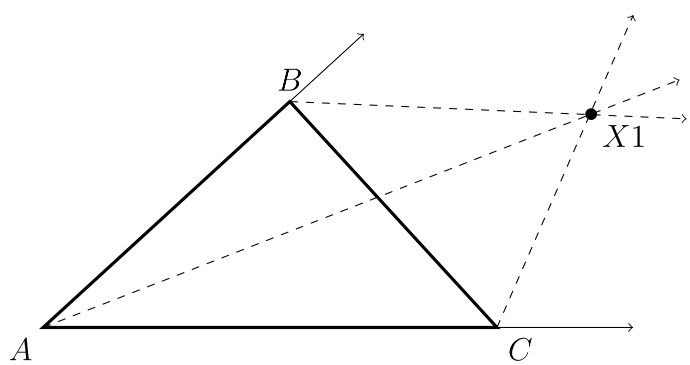
$$\begin{aligned} 37 \times 7 &= 259 \\ 37 \times 8 &= 296 \\ 37 \times 13 &= 481 \end{aligned}$$

The only multiple of 37 with distinct digits whose digit-sum equals the other factor is 481, as $4+8+1 = 13$. Thus $N = 481$.

5. Firstly, let us consider points lying inside $\triangle ABC$. Points lying on the bisector of $\angle A$ are equidistant from AB and AC. Similarly, points lying on the bisector of $\angle B$ are equidistant from AB and BC; points lying on the bisector of $\angle C$ are equidistant from AC and BC. Thus points lying on the intersection of any two of these bisectors are equidistant from all three sides. So the incentre (the common point of intersection of the three angle bisectors) is the only point inside the triangle that is equidistant from ℓ_1, ℓ_2 and ℓ_3.



Now let us consider points lying outside the triangle. We can use a similar argument to show that the common point of intersection between one bisector of the triangle and the bisectors of the exterior angles of the other two vertices of the triangle is also equidistant from ℓ_1, ℓ_2 and ℓ_3. One of these points, marked as $X1$, is shown in the diagram below.



There are three of these points associated with any triangle, and they are called the *excentres* of $\triangle ABC$, as you can draw a circle centred at one of the excentres that is tangent to all three sides of the triangle, extended where necessary.

Interestingly, if we join the three excentres to form a new triangle, DEF say, then $\triangle ABC$ is the orthic triangle of $\triangle DEF$ (the orthic triangle is the triangle obtained by joining the feet of the three altitudes) and the incentre of $\triangle ABC$ is then the orthocentre of $\triangle DEF$.

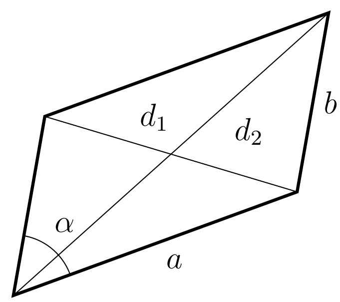
6. If we try the codes

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

then we have tried 1 in every spot, 2 in every spot, 3 in every spot and so on up to 7. If the safe still isn't open this means that the code contains no 1's, no 2's, no 3's and so on up to 7. That is, the code only contains 8s, 9s and 0s, but this isn't enough digits to have 7 distinct digits so the safe must have opened at some point.

Senior Questions

1. (a) Let the acute internal angle of the parallelogram be α.



By the cosine rule,

$$d_1^2 = a^2 + b^2 - 2ab\cos\alpha \quad (1)$$

The other internal angle of the parallelogram is $180° - \alpha$. Thus

$$d_2^2 = a^2 + b^2 - 2ab\cos(180° - \alpha)$$

But $\cos(180° - \alpha) = -\cos\alpha$, hence

$$d_2^2 = a^2 + b^2 + 2ab\cos\alpha \quad (2)$$

Adding equations (1) and (2) together, we obtain

$$d_1^2 + d_2^2 = 2(a^2 + b^2).$$

(b) Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then $\overline{z} = x - iy$ and

$$\begin{aligned} z\overline{z} &= (x+iy)(x-iy) \\ &= x^2 - i^2y^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

Thus

$$\begin{aligned} |z_1+z_2|^2 + |z_1-z_2|^2 &= (z_1+z_2)(\overline{z_1+z_2}) + (z_1-z_2)(\overline{z_1-z_2}) \\ &= (z_1+z_2)(\overline{z}_1+\overline{z}_2) + (z_1-z_2)(\overline{z}_1-\overline{z}_2) \\ &= z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2 \\ &\qquad + z_1\overline{z}_1 - z_1\overline{z}_2 - z_2\overline{z}_1 + z_2\overline{z}_2 \\ &= 2z_1\overline{z}_1 + 2z_2\overline{z}_2 \\ &= 2(|z_1|^2 + |z_2|^2) \end{aligned}$$