MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 15, September 10, 2018

1. If Cog-1 rotates clockwise, Cog-2 must rotate counter clockwise, and so Cog-3 must rotate clockwise and so on.Thus all the odd-numbered cogs must rotate the same way. This means that if Cog-127 is connected to Cog-1, then the cogs cannot be turned.

2. We can work out the side lengths for each square. Let the lower-case letter for a square represent its side length. Then say $b = c - 1$, $g = c - 2$, $f = c - 3$ and so on. With a bit of work we can determine that the total area is 1056.

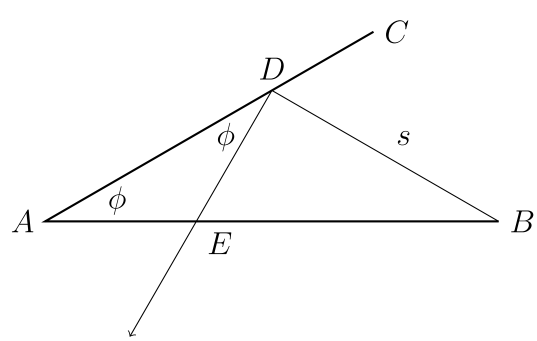
3. Suppose that we are given the length of a side s, the sum of the diagonals, d, and the angle between them, θ.

   (i) Construct a line AB equal to $\frac{d}{2}$.

   (ii) Construct a ray, AC, at an angle of $\phi = \frac{\theta}{2}$ to AB.

   (iii) Using the compasses, find the point D lying on AC which is at a distance s from B.

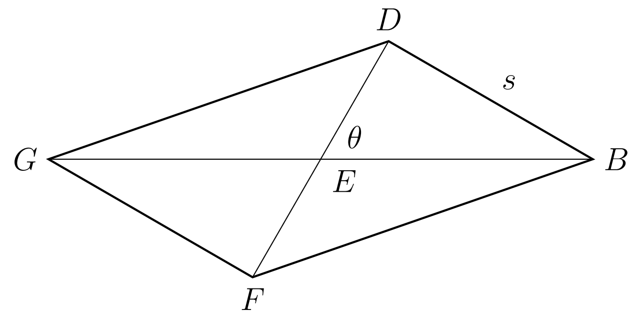
   (iv) Construct a ray DE, also at an angle of phi to AC, that intersects AB at E.



Now since $\angle EAD = \angle EDA$, $\triangle AED$ is isosceles and hence $DE = AE$, which means that $DE + EB = AB = \frac{d}{2}$. Furthermore, by the exterior angle theorem, $\angle DEB = \angle DEA + \angle EAD = \theta$.

(v) Extend DE and BE.

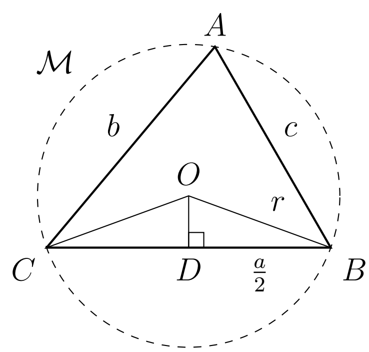
(vi) Using the compasses, find point F on DE such that $EF = DE$, and point G on BE such that $EG = BE$.



Then DF and BG bisect each other and hence $DBFG$ is a parallelogram. Moreover, $DF + BG = d$, the angle between DF and GB is θ, and the length of the side DB is s. Thus $DBFG$ has the required properties.

4. If a number is written in its prime factorisation $n = p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$, then for it to be powerful each of the $m_i \geq 2$ and for it to be a perfect power all $m_i = c$, a constant. Thus for n to be powerful but not a perfect power all the m_i must be greater than 2, but not all the same. The smallest then, would be $2^3 \times 3^2 = 72$.

5. Let O be the centre of $\mathcal{M}$, and let D be the midpoint of BC.



(a) Then $\angle BOC = 2A$, as the angle at the centre is twice the angle at the circumference. Furthermore, $\angle ODB = 90°$, as the perpendicular bisector of a chord passes through the centre. Thus $\triangle BDO$ is a right angled triangle with $\angle BOD = A$, $OB = r$ and $DB = \frac{a}{2}$. Consequently,

$$\begin{aligned}\sin A &= \frac{DB}{OB} \\ &= \frac{a/2}{r}\end{aligned}$$

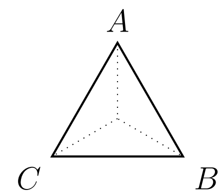
This can be rearranged as $2r = \dfrac{a}{\sin A}$.

(b) We could repeat the previous argument replacing A with B and a with b to show that $2r = \dfrac{b}{\sin B}$. Similarly, it can be shown that $2r = \dfrac{c}{\sin C}$. Thus

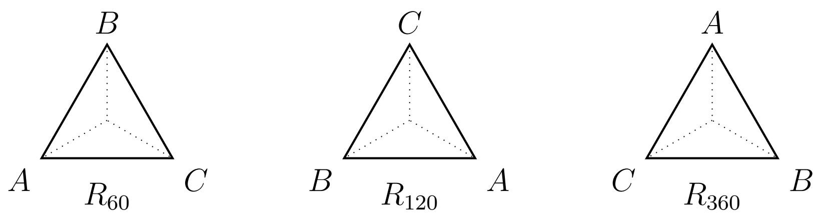
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Senior Questions

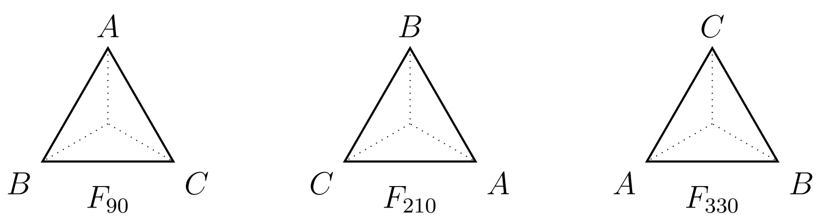
1. (a) Consider the following triangle, which has its vertices labelled A, B, C in a clockwise fashion from the top. We will consider this as the initial position of the triangle.



Then there are three rotations (measured in the counter-clockwise direction), which I will designate R_{60}, R_{120} and R_{360}.

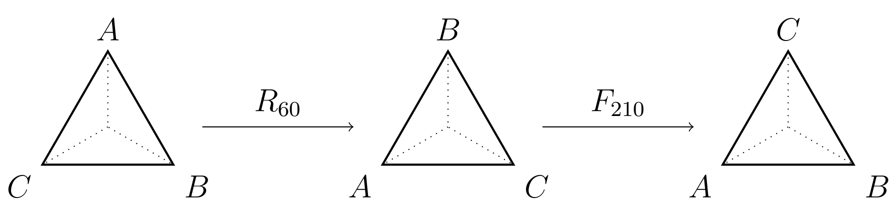


And there are three reflections in the three axes of symmetry of the triangle (flips). I have designated these as F_{90}, F_{210}, and F_{330}, depending on which axis of symmetry is used for the flip.

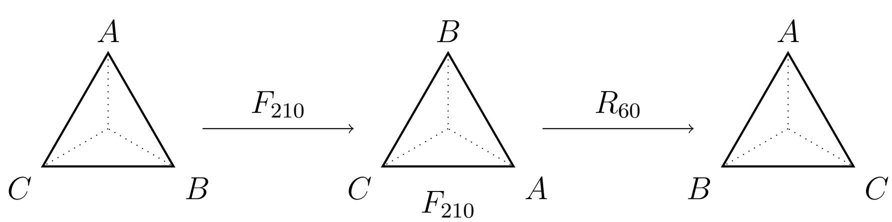


You can also think of these six operations as the six possible permutations of the letters A, B and C.

(b) Consider the following: R_{60} followed by F_{210} is the same as F_{330}.



But F_{210} followed by R_{60} followed by is the same as F_{90}.



Interestingly, there is a subset of the operations that do commute with each other. Can you see which ones they are?

(c) Obviously, this is R_{360}, the "do nothing" operation. (I could also have called it R_0.)

(d) Clearly, R_{360} is it's own inverse, as are the three flipping operations—F_{90}, F_{210} and F_{330}. The two other rotations, R_{60} and R_{120}, are inverses of each other.