MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 15, September 10, 2018

1. If Cog-1 rotates clockwise, Cog-2 must rotate counter clockwise, and so Cog-3 must rotate clockwise and so on. Thus all the odd-numbered cogs must rotate the same way. This means that if Cog-127 is connected to Cog-1, then the cogs cannot be turned.

2. We can work out the side lengths for each square. Let the lower-case letter for a square represent its side length. Then say \( b = c - 1, \ g = c - 2, \ f = c - 3 \) and so on. With a bit of work we can determine that the total area is 1056.

3. Suppose that we are given the length of a side \( s \), the sum of the diagonals, \( d \), and the angle between them, \( \theta \).

   (i) Construct a line \( AB \) equal to \( \frac{d}{2} \).

   (ii) Construct a ray, \( AC \), at an angle of \( \phi = \frac{\theta}{2} \) to \( AB \).

   (iii) Using the compasses, find the point \( D \) lying on \( AC \) which is at a distance \( s \) from \( B \).

   (iv) Construct a ray \( DE \), also at an angle of \( \phi \) to \( AC \), that intersects \( AB \) at \( E \).

Now since \( \angle EAD = \angle EDA \), \( \triangle AED \) is isosceles and hence \( DE = AE \), which means that \( DE + EB = AB = \frac{d}{2} \). Furthermore, by the exterior angle theorem, \( \angle DEB = \angle DEA + \angle EAD = \theta \).
(v) Extend $DE$ and $BE$.

(vi) Using the compasses, find point $F$ on $DE$ such that $EF = DE$, and point $G$ on $BE$ such that $EG = BE$.

Then $DF$ and $BG$ bisect each other and hence $DBFG$ is a parallelogram. Moreover, $DF + BG = d$, the angle between $DF$ and $GB$ is $\theta$, and the length of the side $DB$ is $s$. Thus $DBFG$ has the required properties.

4. If a number is written in its prime factorisation $n = p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$, then for it to be powerful each of the $m_i \geq 2$ and for it to be a perfect power all $m_i = c$, a constant. Thus for $n$ to be powerful but not a perfect power all the $m_i$ must be greater than 2, but not all the same. The smallest then, would be $2^3 \times 3^2 = 72$.

5. Let $O$ be the centre of $\mathcal{M}$, and let $D$ be the midpoint of $BC$.

(a) Then $\angle BOC = 2A$, as the angle at the centre is twice the angle at the circumference. Furthermore, $\angle ODB = 90^\circ$, as the perpendicular bisector of a chord passes through the centre. Thus $\triangle BDO$ is a right angled triangle with $\angle BOD = A$, $OB = r$ and $DB = \frac{a}{2}$. Consequently,

$$\sin A = \frac{DB}{OB} = \frac{a/2}{r}$$

This can be rearranged as $2r = \frac{a}{\sin A}$.

(b) We could repeat the previous argument replacing $A$ with $B$ and $a$ with $b$ to show that $2r = \frac{b}{\sin B}$. Similarly, it can be shown that $2r = \frac{c}{\sin C}$. Thus

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$
Senior Questions

1. (a) Consider the following triangle, which has its vertices labelled $A$, $B$, $C$ in a clockwise fashion from the top. We will consider this as the initial position of the triangle.

Then there are three rotations (measured in the counter-clockwise direction), which I will designate $R_{60}$, $R_{120}$ and $R_{360}$.

And there are three reflections in the three axes of symmetry of the triangle (flips). I have designated these as $F_{90}$, $F_{210}$, and $F_{330}$, depending on which axis of symmetry is used for the flip.

You can also think of these six operations as the six possible permutations of the letters $A$, $B$ and $C$.

(b) Consider the following: $R_{60}$ followed by $F_{210}$ is the same as $F_{330}$.

But $F_{210}$ followed by $R_{60}$ followed by is the same as $F_{90}$.
Interestingly, there is a subset of the operations that do commute with each other. Can you see which ones they are?

(c) Obviously, this is $R_{360}$, the “do nothing” operation. (I could also have called it $R_{0}$.)

(d) Clearly, $R_{360}$ is it’s own inverse, as are the three flipping operations—$F_{90}$, $F_{210}$ and $F_{330}$. The two other rotations, $R_{60}$ and $R_{120}$, are inverses of each other.