MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 17, September 24, 2018

1. Firstly, we calculate the prime factorisation of N.

$$\begin{aligned} N &= 1^9 \times 2^8 \times 3^7 \times 4^6 \times 5^5 \times 6^4 \times 7^3 \times 8^2 \times 9^1 \\ &= 2^8 \times 3^7 \times 2^{12} \times 5^5 \times (2 \times 3)^4 \times 7^3 \times 2^6 \times 3^2 \\ &= 2^{30} \times 3^{13} \times 5^5 \times 7^3 \end{aligned}$$

A divisor of N that is a perfect square has a prime factorisation with all primes raised to an even number. Thus there are 16 choices for the number of 2's $(0, 2, 4, \ldots, 30)$; 7 choices for the number of 3's; 3 choices for the number of 5's; and 2 choices for the number of 7's. The total is $16 \times 7 \times 3 \times 2 = 672$.

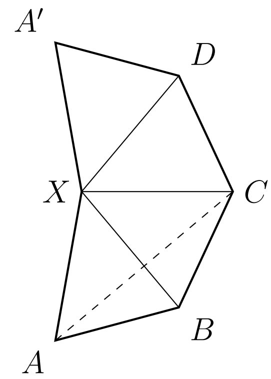
2. Firstly, note that neither b nor c can be zero. Then simplifying the given equation, we have

$$\begin{aligned} \frac{b}{b} \times \frac{a/b}{c} &= \frac{a}{b/c} \times \frac{c}{c} \\ \frac{a}{bc} &= \frac{ac}{b} \\ ab &= abc^2 \\ ab(1 - c^2) &= 0 \end{aligned}$$

This equation has the solutions $a = 0$, $b = 0$ or $c = \pm 1$. As noted previously, $b \neq 0$. So we are left with $a = 0$ or $c = \pm 1$.

We now calculate the number of triplets, making sure not to inadvertently double-count any. If $a = 0$, then there are 20 choices for b and c, as $b, c \neq 0$. Thus there are 400 triplets with $a = 0$. If $c = 1$, then there are 20 choices for a, as the $a = 0$ case has already been counted in the previous step, and also 20 choices for b. This gives us another 400 triplets. Similarly, if $c = -1$, then there are 400 triplets. Thus there are 1200 triplets altogether.

3. Imagine that we detach the pyramid from its base, cut along AX and spread the top of the pyramid out flat. We would get a (possibly convex) polygon like that shown below.



The shortest distance from A to C is the straight line shown by the dotted path. We know that $AB = BC = b$. Let the length of AX be s. Then $AX = BX = CX = s$ (all the faces of the pyramid are isosceles triangles). Let $\angle ABX = \theta$. By the cosine rule,

$$\begin{aligned}\cos\theta &= \frac{b^2 + s^2 - s^2}{2bs} \\ &= \frac{b}{2s}.\end{aligned}$$

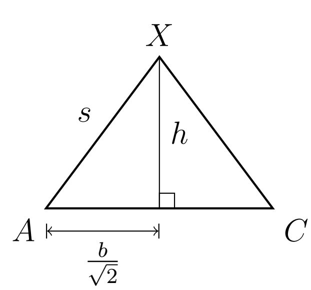
Applying the cosine rule to $\triangle ABC$, we have

$$\begin{aligned}AC^2 &= 2b^2 - 2b^2\cos(2\theta) \\ &= 2b^2[1 - \cos(2\theta)].\end{aligned}$$

Using the trigonometric identity $\cos(2\theta) = 2\cos^2\theta - 1$, we have

$$\begin{aligned}AC^2 &= 4b^2(1 - \cos^2\theta) \\ &= 4b^2\left(1 - \frac{b^2}{4s^2}\right).\end{aligned}$$

To determine s as a function of b and h, consider a vertical slice through the pyramid that runs along one of the diagonals of the square base. Since the base is a square with sides of length b, the length of the diagonal is $\sqrt{2}b$.



So by Pythagoras' theorem

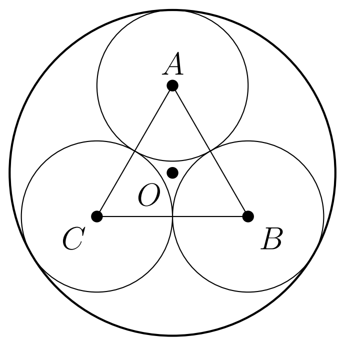
$$s^2 = h^2 + \frac{b^2}{2}.$$

Thus

$$AC^2 = 4b^2\left(1 - \frac{b^2}{4h^2 + 2b^2}\right)$$
$$\therefore AC = 2b\sqrt{1 - \frac{b^2}{4h^2 + 2b^2}}.$$

Since $\sqrt{1 - \frac{b^2}{4h^2+2b^2}} < 1$, we can see that the ant will always walk a shorter distance if it goes over the pyramid rather than around the base.

4. Let the centres of the small circles be A, B and C, and the centre of the big circle be O.



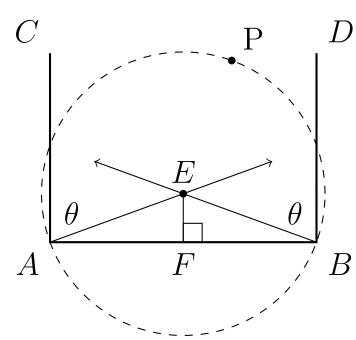
Then $\triangle ABC$ is isosceles with side length $2r$, and medians of length $\sqrt{3}r$. The centroid of $\triangle ABC$ is the centre of the big circle, O, and is located 2/3 of the way along the median. Thus $OA = \frac{2\sqrt{3}r}{3}$. So $R = r + \frac{2\sqrt{3}r}{3} = r\left(1 + \frac{2}{\sqrt{3}}\right)$.

5. (a) If θ is acute, then $\mathcal{M}$ can be constructed as follows:

   (i) Construct an interval, AB, of length b.

   (ii) Raise perpendiculars AC and BD to AB at the points A and B, as shown in the diagram.

   (iii) Construct rays at an angle of θ to AC and DB at the points A and B, respectively. Let the point of intersection of these rays be E.

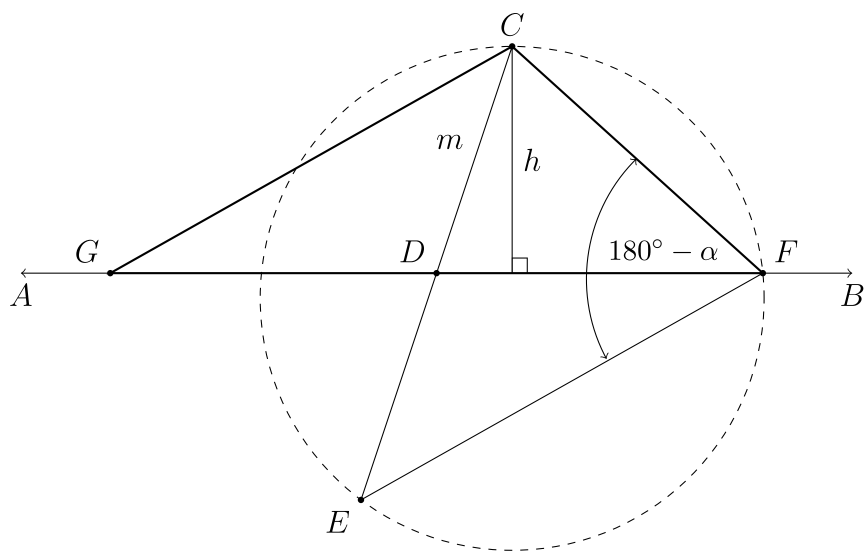


The centre of the required circumcircle is E and AE is the radius.

Proof: Let F be the foot of the perpendicular from E to AB. Then $AC||EF$. Consequently, $\angle AEF = \angle EAC = \theta$, as these are alternate angles. Similarly, $\angle DBE = \angle BEF = \theta$. Thus $\triangle AEF$ is similar to $\triangle BEF$ by AAS, which implies that $AE = EB$. Thus a circle centred at E with radius AE will also pass through the point B. Furthermore, $\angle AEB = 2\theta$, so if P is any point on the major arc of the circle cut off by the chord AB, then $\angle APB = \theta$, as the angle at the circumference is half the angle at the centre. So the vertex lies somewhere on the major arc.

This construction fails if $\theta = 90°$, as there is no unique point of intersection of the two rays AE and BE. However, in this case, AB is the diameter of $\mathcal{M}$ (as a consequence of Thales' theorem), and the construction is simple. If θ is obtuse, we can repeat the procedure outlined above, with the proviso that P will now lie on the minor arc of the circle. Some minor details of the proof will differ.

(b) Suppose that the angle at the vertex is α, the altitude has height h and the median length m.



(i) Draw a line AB. (The base of the triangle will be some segment of this line.)

(ii) Raise a perpendicular to AB with length h. Let C be the endpoint of this perpendicular that does not lie on AB.

(iii) Using the compasses, find the point D lying on AB such that $CD = m$.

(iv) Extend CD until it has length $2m$. Designate this point E.

(v) Construct a circle that has CE as a chord and such that any angle lying on the major arc of this circle and subtending CE is equal to $180° - \alpha$. (This can be done using the construction method from (a).)

(vi) Find F, the point of intersection between the circle and AB, produced if necessary.

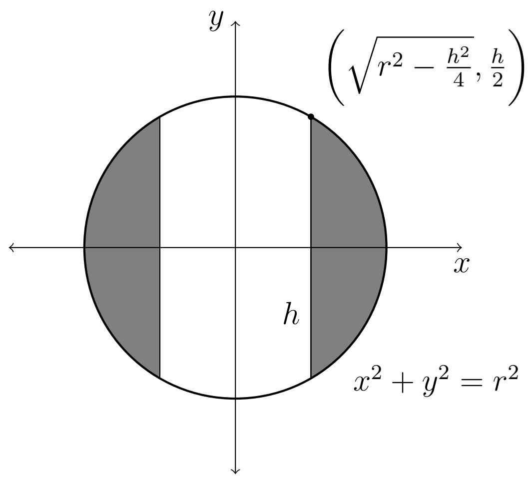
(vii) Using the compasses, find the point G lying on AB such that $DG = DF$.

The required triangle is $\triangle CFG$.

Proof: Clearly, $\triangle CFG$ has altitude h. Since D is the midpoint of GF, it also has median of length m. Now consider the quadrilateral $CFEG$. This quadrilateral has diagonals GF and CE which bisect each other. Hence $CFEG$ is a parallelogram. Thus $CG||EF$ and so $\angle FCG$ and $\angle CFE$ are co-interior. Thus $\angle GCF = \alpha$, as required.

Senior Questions

1. We calculate the volume by integration using circular shells.



The volume, V is given by

$$V = 2\pi \int_a^b xy\,dx$$

In this case,

$$\begin{aligned} y &= \sqrt{r^2 - x^2} - -\sqrt{r^2 - x^2} \\ &= 2\sqrt{r^2 - x^2}. \end{aligned}$$

The limits of integration are $a = \sqrt{r^2 - \frac{h^2}{4}}$ (from Pythagoras' theorem) and $b = r$. Thus

$$\begin{aligned} V &= 2\pi \int_{\sqrt{r^2 - \frac{h^2}{4}}}^{r} 2x\sqrt{r^2 - x^2}\,dx \\ &= 2\pi \left[-\frac{2}{3}(r^2 - x^2)^{3/2} \right]_{\sqrt{r^2 - \frac{h^2}{4}}}^{r} \\ &= \frac{4\pi}{3} \left[-(r^2 - x^2)^{3/2} \right]_{\sqrt{r^2 - \frac{h^2}{4}}}^{r} \\ &= \frac{4\pi}{3} \left(\frac{h^2}{4} \right)^{3/2} \\ &= \frac{\pi h^3}{6} \end{aligned}$$

2. (a) Note that if $|x| < 1$ then the RHS is a convergent geometric series with $a = 1$ and common ratio $-x$. Thus

$$1 - x + x^2 - x^3 + \ldots = \frac{1}{1 - (-x)} = \frac{1}{1 + x}$$

Integrating both sides of the equation with respect to x (which is valid for a convergent infinite series), we have

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots,$$

where C is the constant of integration. To evaluate C, we substitute $x = 0$. Then we have $C = 0$. Thus

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots.$$

(b) From the calculator, $\ln(1.1) \approx 0.09531018$. If we substitute $x = \frac{1}{10}$ into the formula we derived in part (a),

Terms	Value
1	0.1
2	0.095
3	0.09533333
4	0.09530833
5	0.09531033
6	0.09531017

As you can see, we get approximately one more digit of accuracy for each extra term we add, so to get five decimal places, we need to take five terms in the sequence.