MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 2, May 14, 2018

1. If we take the prime factorisation of the numbers 2 through 10, we get
$$2, 3, 2^2, 5, 2 \times 3, 7, 2^3, 3^2, 2 \times 5.$$
So the smallest number divisble by all these numbers is the number with prime factorisation made by taking as few as these as possible so that each factorisation above is included: $2^3 \times 3^2 \times 5 \times 7 = 2520$ gives our answer.

2. (a) Assuming $n > 0$ is positive
$$n^2 + n < n^2 + 2n + 1 = (n + 1)2$$
and
$$n^2 < n^2 + n$$
So $n^2 < n^2 + n < (n + 1)2$ so $n^2 + n$ can't be a square. If $n < -1$,
$$n^2 + 2n + 1 < n^2 + n < n^2,$$
so similarly $n2 + n$ is not a square.

   (b) Note that, if $n > 1$,
$$\left(n^2 + \frac{n}{2}\right)^2 = n^4 + n^3 + \frac{n^2}{4} < n^4 + n^3 + n^2 + n,$$
and
$$\left(n^2 + \frac{n}{2} - 1\right)^2 = n^4 + n^3 + \frac{9n^2}{4} + n + 1 > n^4 + n^3 + n^2 + n.$$

3. Let the isoceles triangle be ABC with base BC. The square is bisected by the alitude of the triangle through A, which meets BC at D. Let E be the vertex of the square on BC between B and D and let F be the vertex of the square above E. Then triangles ABD and FBE are similar, so, letting the side length of the square be x, we get the relation
$$\frac{x}{\sqrt{10^2 - 6^2}} = \frac{6 - x/2}{6},$$
the solution of which is $x = 4.8$.

4. We wish to

   find n, such that for some q_1, q_2, q_3 and r we have

$$\begin{aligned}364 &= nq_1 + r\\ 414 &= nq_2 + r\\ 539 &= nq_3 + r\end{aligned}$$

   Combining the

   first two means

$$(q_2 - q_1)n = 414 - 364 = 50$$

   Since n and all the q's are integers, n must be a factor of 50, which are 50, 25, 10, 5, 2 and 1. Dividing 364 or 414 by 50 gives a remainder of 14, whilst dividing 539 by 50 gives a remainder of 39, so n is not 50. Dividing 364 or 414 by 25 still gives a remainder of 14, and so does dividing 539 by 25. So $n = 25$ works, and since it is larger than the other factors of 50, it is our answer.

5. The point P lies on the opposite side of the chord from O. Then $\angle APB = 120°$.

6. Let the two types of coins be A and B. Split the 128 coins into two piles of 64 each. Now we weigh the two piles. If they are the same weight, this means that both piles have the same number of A coins, so we discard one of the piles, and split the remainder into two equal piles. If we have the good fortune that our two piles are always of equal weight, then after 6 weighings we have 2 coins left, and they must be of different type.

   Suppose now that at some point the two piles are not of equal weight. Now take half the coins from each pile and weigh them. If they are equal in weight, then discard these coins and continue with the others. If they are different in weight, discard the others and continue with these coins.

   We will show that there is always at least one of each type remaining. Suppose at any step, n type A coins remain. We show that it is impossible to remove all n coins. If the two new piles are even in weight, then we can only remove $\frac{n}{2}$. If they are uneven in weight then we can remove at most all but 1 type A coin (if we happen to only select 1 A coin for the second weighing).

Senior Questions

1. We are given

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}. \qquad (1)$$

   Multiplying both sides by $\frac{1}{4}$,

$$\frac{1}{4(1)^2} + \frac{1}{4(2)^2} + \frac{1}{4(3)^2} + \frac{1}{4(4)^2} + \ldots = \frac{\pi^2}{24}.$$

   So

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \ldots = \frac{\pi^2}{24}. \qquad (2)$$

Subtracting (2) from (1), we have

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}.$$

Let

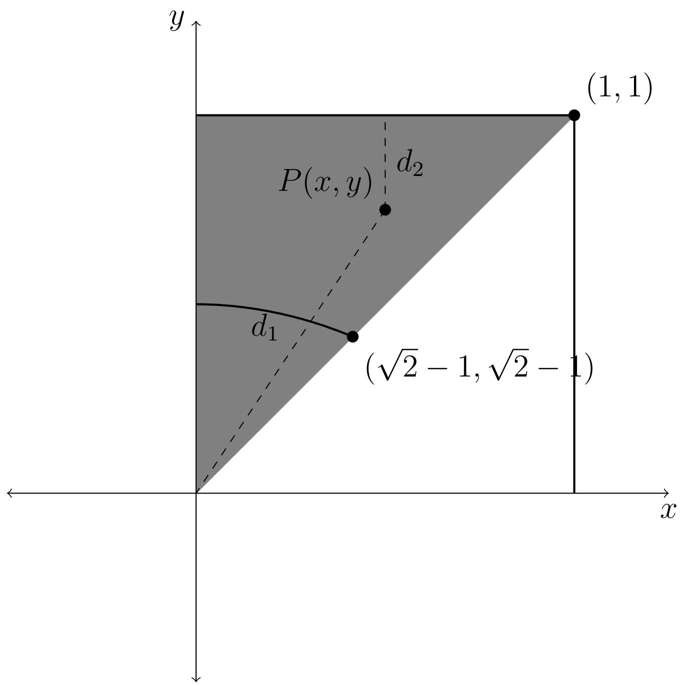
$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = x \qquad (3)$$

Adding (1) and (3), we have

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right) = \frac{\pi^2}{6} + x$$

$$\therefore x = \frac{\pi^2}{4} - \frac{\pi^2}{6} = \frac{\pi^2}{12}$$

2. Consider the target as a square in the Cartesian number plane, with the bullseye at the origin and corners at $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$. Let the point that the arrow strikes the target be $P(x,y)$, where $-1 \leq x, y \leq 1$. Let d_1 be the distance from P to the origin, and let d_2 the distance to the nearest edge. By symmetry, we may restrict our attention to shaded triangle that represents the upper half of the part of the target that lies in the first quadrant, as shown.



We need to calculate the what fraction of the triangle has $d_1 \leq d_2$. Clearly, $d_1 = \sqrt{x^2+y^2}$, and the nearest edge is the line $y = 1$, so $d_2 = 1 - y$. Thus if $d_1 = d_2$,

$$\begin{aligned}(1-y)^2 &= x^2 + y^2 \\ 1 - 2y + y^2 &= x^2 + y^2 \\ y &= \frac{1}{2}(1 - x^2)\end{aligned}$$

This parabola crosses the line $y = x$ at the point $(\sqrt{2}-1, \sqrt{2}-1)$. As the total area of the triangle is $\frac{1}{2}$, the required fraction, A, is given by

$$\begin{aligned} A &= 2\int_0^{\sqrt{2}-1} \frac{1}{2}(1-x^2) - x\,dx \\ &= \int_0^{\sqrt{2}-1} 1 - x^2 - 2x\,dx \\ &= \frac{(4\sqrt{2}-5)}{3} \approx 0.219 \end{aligned}$$