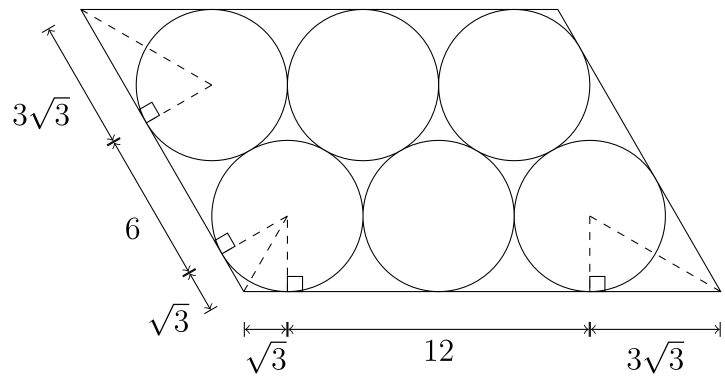
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 3, May 28, 2018

1. The dimensions of the brick are integers L, W and H with $L + W = 9$ cm and $LWH = 42$ cm^3. This implies that $H = 42/(L \times W)$ cm. Only $L = 2$, $W = 7$ has LW divide 42, and so $H = 3$ cm.

2. A four digit palindromic number x has the form $ABBA$. That is, $x = 1001A + 110B$. Now $1001 = 7 \times 143$, but $110 = 11 \times 10$, which is not a multiple of 7. Consequently, a four-digit palindromic number that is divisible by 7 has the form $A00A$ or $A77A$, where A can be any of the numbers $1, 2, \ldots, 9$. Thus there are 18 such numbers.

3. The internal angles of the parallelogram are 60° and 120°. Using trigonometry, it can be shown that the base of the parallelogram has length $12 + 4\sqrt{3}$ cm and the side has length $6 + 4\sqrt{3}$ cm. Thus the area is $12(9 + 5\sqrt{3})$ square centimetres.



4. (a) Since $a + b + c = 2$ and $a + b > c$, $a + c > b$ and $b + c > a$ each of a, b and c is less than one.

   (b)

$$
\begin{aligned}
(1-a)(1-b)(1-c) &> 0 \\
1 - (a+b+c) + ab + bc + ca - abc &> 0 \\
-1 + ab + bc + ca - abc &> 0
\end{aligned}
$$

and

$$\begin{aligned}(a+b+c)^2 &= 4\\ a^2+b^2+c^2+2(ab+bc+ca) &= 4\\ ab+bc+ca &= 2-\frac{1}{2}(a^2+b^2+c^2)\end{aligned}$$

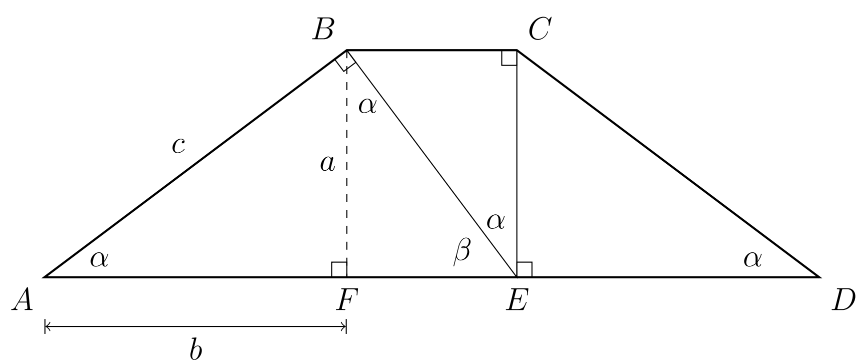
Combining the two yields the answer.

5. (a) $x_0 = 0$, $x_1 = 1$, $x_2 = 1$, $x_3 = 3$, $x_4 = 5$, $x_5 = 11$, $x_6 = 21$.
   (b) Validate by substituting into the recursive rule $x_{n+1} = x_n + 2x_{n-1}$ and confirming that the two initial conditions are satisfied.
   (c) Consider the sequence in mod 3.

Senior Questions

1. Let $p(x) = (3+2x+x^2)^{2018} = a_0 + a_1x + a_2x^2 + \ldots + a_{4036}x^{4036}$.
   (a) $a_0 = p(0) = 3^{2018}$ and $a_1 = p'(0) = (2018)(3+2(0)+(0)^2)^{2017}(2+2\times 0) = (2)(2018)(3)^{2018}$.
   (b) $a_0 + a_1 + a_2 + \ldots + a_{4036} = p(1) = 6^{2018}$
   (c) $a_0 - a_1 + a_2 - a_3 + \ldots + a_{4036} = p(-1) = 2^{2018}$

2. Firstly, after some trial and error, we arrive at the following diagram.



Let F be the foot of the perpendicular from B to AD.

Let $BF = a$, $AF = b$ and $AB = c$.

Let $\angle BAF = \alpha$ and $\angle BEA = \beta$. (With a little angle chasing, it can be shown that the other angles are as in the diagram. Also note that $\triangle ABF \equiv \triangle DCE$.)

Note that $\angle ABE$, $\angle BCE$ and $\angle CED$ are right angles, and that a, b and c are integers, with

$$a^2 + b^2 = c^2.$$

We must find the length $BC = EF$, given that $AD = 2009$.

From $\triangle ABF$, we can see that $\tan\alpha = \dfrac{a}{b}$. Similarly, from $\triangle BEF$ we have $\tan\alpha = \dfrac{EF}{a}$, so $EF = a\tan\alpha = \dfrac{a^2}{b}$.

Since $AD = 2009$, we have,

$$2b + \frac{a^2}{b} = 2009$$
$$2b^2 + a^2 = 7^2.41b.$$

Seeing that $a^2 + b^2 = c^2$, and that 41, the only prime factor of 2009 congruent to 1 mod(4), can be written uniquely as the sum of two squares, we may re-write this last equation as

$$b^2 + c^2 = 7^2 b(16 + 25).$$

As c is the hypotenuse of $\triangle AFB$, $c > b$, so we can conclude that $b^2 = 16.7^2 b$ and $c^2 = 25.7^2 b$. Hence $b = 4^2.7^2 = 784$, $c = 5.4.7^2 = 980$, $a = 3.4.7^2 = 588$. (That is, $\triangle ABF$ has sides in the ratio $3 : 4 : 5$.)

Thus $BC = \dfrac{a^2}{b} = \dfrac{3^2.4^2.7^4}{4^2.7^2} = 441$.