1. Since $x$ is an integer, $x^2$ is the product of even powers of 2 and 3, and hence $y^3$ is also a product of even powers of 2 and 3. Then $y^3$ can be $1$, $2^6$, $2^{12}$, $3^6$, $3^{12}$, $2^6 \cdot 3^6$, $2^6 \cdot 3^{12}$, $2^{12} \cdot 3^6$ or $2^{12} \cdot 3^{12}$. For each of these $y$ values, there is one value of $x$. Hence there are nine solutions altogether.

2. Write $\frac{11}{42}$ as a simple continued fraction. That is,

\[
\frac{11}{42} = \frac{1}{\frac{42}{11}} = \frac{1}{3 + \frac{9}{11}} = \frac{1}{3 + \frac{1}{\frac{11}{9}}} = \frac{1}{3 + \frac{1}{1 + \frac{2}{9}}}
\]

Then $a + b + c + d = 3 + 1 + 4 + 2 = 10$.

3. We use the method of reflection.

Let $A$ be a point lying inside the angle $XOY$ and let $B$ and $C$ be points on $OX$ and $OY$ as shown in the diagram. Let $D$ and $E$ be the reflection of the point $A$ in the lines $OX$ and $OY$, respectively. Then $\triangle ABD$ and $\triangle ACE$ are both isosceles with $AC = CE$ and $AB = BD$. Thus the path from $D$ to $E$ via $B$ and $C$ is equal in length to the perimeter of $\triangle ABC$. Hence this length is minimised when $DBCE$ is a straight line.
4. The sum of the digits 1, 2, 3, . . . , 9 is 45 [(1 + 9) + (2 + 8) + . . . + 5]. Also recalling that if we have a sum like \( \sum_{k=0}^{n} (a + k) = a(n + 1) + \sum_{k=0}^{n} k \), then the required sum is

\[
\sum_{a=0}^{9} \sum_{b=0}^{9} \sum_{c=0}^{9} \sum_{d=0}^{9} (a + b + c + d) = \sum_{a=0}^{9} \sum_{b=0}^{9} \sum_{c=0}^{9} \left( 10(a + b + c) \sum_{d=0}^{9} d \right) \\
= \sum_{a=0}^{9} \sum_{b=0}^{9} \sum_{c=0}^{9} (45 + 10a + 10b + 10c) \\
= \sum_{a=0}^{9} \sum_{b=0}^{9} \left( 10(45 + 10a + 10b) + 10 \sum_{c=0}^{9} c \right) \\
= \sum_{a=0}^{9} \sum_{b=0}^{9} (450 + 100a + 100b + 450) \\
= \sum_{a=0}^{9} \left( 10(900 + 100a) + 100 \sum_{b=0}^{9} b \right) \\
= \sum_{a=0}^{9} (9000 + 1000a + 4500) \\
= 10 \times 13500 + 1000 \times 45 \\
= 180000
\]

Senior Questions

1. Firstly, we complete the square in a slightly unusual way.

\[
x^2 - 19x + 94 = x^2 - 20x + 100 + x - 6 \\
= (x - 10)^2 + x - 6
\]

Then \((x - 10)^2\) is a perfect square whenever \(x\) is an integer.

Consider the following diagram

\[
\begin{array}{c|c}
\hline
x - 10 & y \\
\hline
\end{array}
\]

We want to make

\[
(x - 10 + y)^2 = x^2 - 20x + 100 + x - 6,
\]
where $x$ and $y$ are integers. Thus

\[
y^2 + 2(x - 10)y = x - 6 \\
y^2 + 2xy - 20y = x - 6 \\
y^2 - 20y + 6 = x(1 - 2y)
\]

So

\[
x = \frac{y^2 - 20y + 6}{1 - 2y}.
\]

Using polynomial long division, we find that

\[
x = -\frac{y}{2} + \frac{39}{4} - \frac{15}{4} \left( \frac{1}{1 - 2y} \right).
\]

We multiply this by 4 to obtain

\[
4x = -2y + 39 - \frac{15}{1 - 2y}.
\]

This can be made simpler if we re-write it as

\[
4x = 1 - 2y + 38 - \frac{15}{1 - 2y},
\]

and then make the substitution $w = 1 - 2y$, so then

\[
4x = w - \frac{15}{w} + 38.
\]

If we want to have $x$ an integer, then $w$ must be a factor of 15. Since there are a finite number of integer solutions for $w$ ($\pm 1, \pm 3, \pm 5, \pm 15$), we simply need to find the one that gives the largest value of $x$. If we do this, we find that $x = 13$.

2. We use the method of reflection again. Let $B'$ be the reflection of the point $B$ in the river. Then the length $L$ is equal to the path from $A$ to $B'$ via $E$, which is minimized when $AEB'$ is a straight line. In this case, the distance is 15 km (a nice 3-4-5 right triangle).