MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 5, June 11, 2018

1. Simplifying $(a+b)^2 - (a-b)^2 > 29$, we obtain $4ab > 29$. Thus the smallest value of $4ab$ is 32, in which case, $ab = 8$, and the smallest value of a is 4.

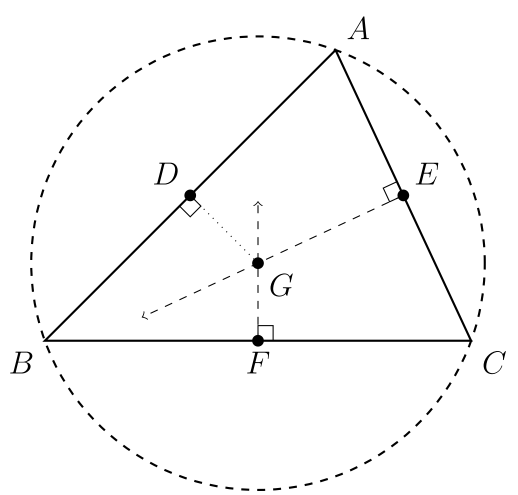
2. If we substitute $y = x + c$ into $x^2 + y^2 = 1$, we obtain the quadratic equation

$$x^2 + cx + \frac{c^2 - 1}{2} = 0.$$

If there is only one solution, we must have $\Delta = 0$. Thus

$$\begin{aligned} c^2 - 2(c^2 - 1) &= 0 \\ c &= \pm\sqrt{2} \end{aligned}$$

3. Let E and F be the midpoints of sides AC and BC, as shown in the diagram. Let perpendiculars from E and F intersect at G. Let DG be a perpendicular from G to side AB. We need to show that D is also the mid-point of AB.



Since EG and FG are the perpendicular bisectors of AC and BC, AC and BC can be considered chords of a circle centred at G. But then AB is also a chord on the same circle, and since DG is a perpendicular from the centre of the circle to the chord, it bisects AB. Thus D is the mid point of AB, as required.

4. Letting $a = \sqrt[3]{5\sqrt{13} + 18}$ and $b = \sqrt[3]{5\sqrt{13} - 18}$, $x = a - b$ then we find that, after expanding $(a-b)^3$

$$\begin{aligned} (a-b)^3 &= a^2 - 3a^2b + 3ab^2 - b^3 \\ &= a^3 - b^3 - 3ab(a-b). \end{aligned}$$

Now $a^3 - b^3 = 36$ and $ab = 1$. Thus

$$x^3 = 36 - 3x,$$

which has the solution $x = 3$.

5. We have $x^2 - 8x - 1001y^2 = 0$, so

$$y^2 = \frac{x(x-8)}{1001} = \frac{x(x-8)}{7 \cdot 11 \cdot 13}$$

Now $x = 0$ and $x = 8$ are not permitted.
Checking:

$y = 1$: Then $x(x-8) = 7 \cdot 11 \cdot 13$, which is not possible.

$y = 2$: Then $x(x-8) = 4 \times 7 \cdot 11 \cdot 13$, which is also not possible.

$y = 3$: Then $x(x-8) = 9 \times 7 \cdot 11 \cdot 13 = 99 \times 91$, so $x = 99$ and $y = 2$ thus the smallest value of $x + y$ is 102.

Senior Questions

1. Since $g(-x) = -g(x)$ for all x in the domain, if $x = 0$ is in the domain, then

$$g(-0) = -g(0).$$

But $g(-0) = g(0)$, so this is only possible if $g(0) = 0$.

2. (a) Use the chain rule and the definition of an even function.
   (b) Again, use the chain rule.

3. $f(x) = \frac{1}{2}[h(x) + h(-x)]$ and $g(x) = \frac{1}{2}[h(x) - h(-x)]$.

4. Yes, the zero polynomial, $z(x) = 0$, is both odd and even.