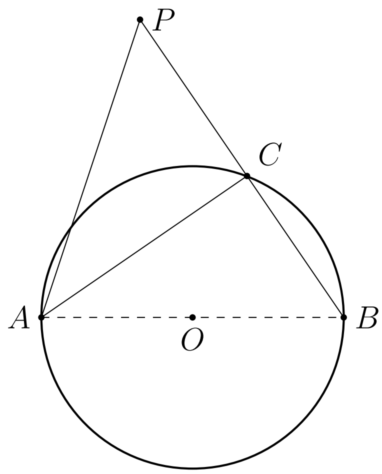
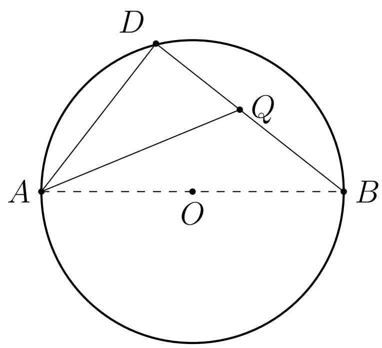
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 7, June 25, 2018

1. Suppose that PB intersects the circle at C, as shown.



Then, by Thales' theorem, $\angle ACB = 90°$, and consequently $ACP = 90°$. Thus $\triangle ACP$ is a right angle triangle, with $\angle APC$ being one of the smaller angles, hence it is acute.

Now consider the point Q, which lies inside the circle. This time, extend BQ until it intersects the circle at D.



Once again, $\angle ADB = 90°$, and by the exterior angle theorem, $\angle AQB = \angle ADB + \angle DAQ > 90°$.

2. (a) Note: this question mshould have read: "Explain why, if a^2+b^2 has a fixed value, ab is greatest when $a=b$."

For all real numbers a and b,

$$\begin{aligned} 0 &\leq (a-b)^2 \\ 0 &\leq a^2 - 2ab + b^2 \\ ab &\leq \frac{a^2+b^2}{2} \end{aligned}$$

Furthermore, equality only holds if $a=b$.

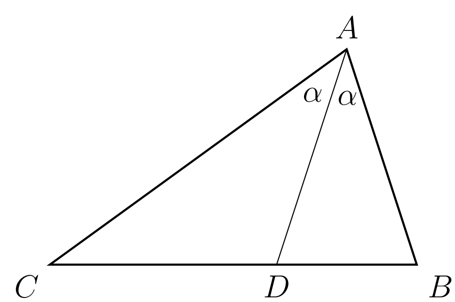
(b)

$$\begin{aligned} x^4+y^4 &= x^4 + 2x^2y^2 + y^4 - 2x^2y^2 \\ &= (x^2+y^2)^2 - 2x^2y^2 \\ &= c^4 - 2(xy)^2 \end{aligned}$$

This quantity is minimised when xy is maximised. That is, when $x=y$. If $x=y$, then $x^2=c^2/2$, in which case

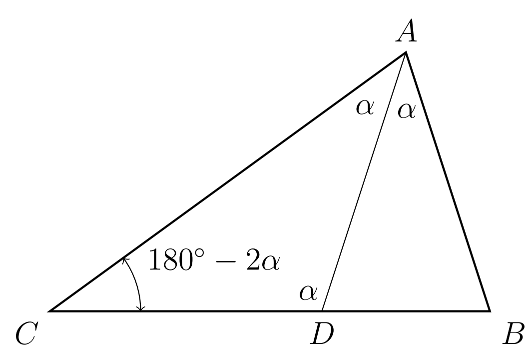
$$x^4+y^4 = c^4 - 2\times\left(\frac{c^2}{2}\right)^2 = \frac{c^4}{2}$$

3. Suppose that ABC is a triangle. AD is an angle bisector, and $\triangle ADB$ and $\triangle ADC$ are both isosceles. Let $\angle CAD = \alpha$. Then $\angle DAB = \alpha$. Consider $\triangle ACD$, which is isosceles. There are three possibilities:

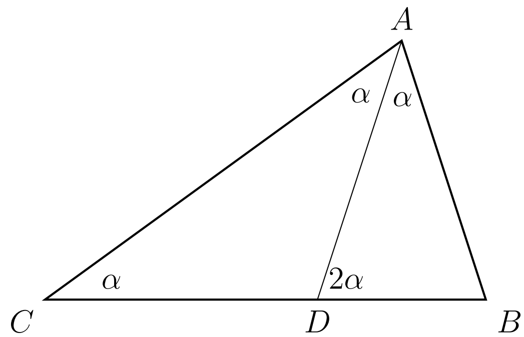


(a) $\angle ACD$ is the vertex;
(b) $\angle ADC$ is the vertex; or
(c) $\angle CAD$ is the vertex.

If $\angle ACD$ is the vertex then $\angle CAD = \angle ACD = \alpha$ and thus $\angle ACD = 180° - \alpha$. However, if we consider the angle sum of $\angle ABC$, this implies that $\angle ABC = 0°$. In which case, the triangle is degenerate.



If $\angle ADC$ is the vertex then $\angle ACD = \angle DAC = \alpha$ and thus $\angle ADB = \angle ACD + \angle CAD$, since it is the external angle of $\angle ADC$. So $\angle ADB = 2\alpha$. Since $\triangle ADB$ is also isosceles, it also has a pair of equal angles, and so $\angle ABD = \alpha$ or $\angle ABD = 2\alpha$.



If $\angle ABD = \alpha$, then by the angle sum of $\triangle ABD$, we have $4\alpha = 180°$, so $\alpha = 45°$. Hence $\angle A = 90°$, $\angle B = \angle C = 45°$. If, however, $\angle ABD = 2\alpha$, then by the angle sum of $\triangle ABD$, we have $5\alpha = 180°$, so $\alpha = 36°$. Hence $\angle A = \angle B = 72°$, $\angle C = 36°$.

If $\angle CAD$ is the vertex, than we have two cases that are essentially the same as those above, only with the orientations of the smaller triangles reversed. Hence we get the same two solutions.

4. Suppose a monic quadratic is factorised with roots α and β. Then

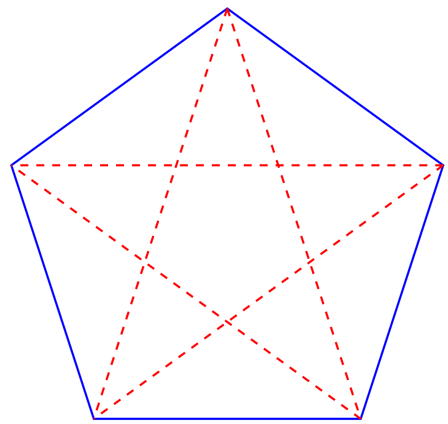
$$(x-\alpha)(x-\beta) = x^2 - (a+b)x + ab$$

Since 2 343 643 is odd, one of α and β must be odd and the other even, but the product of an odd and even number is even, and so cannot be 2 382 987. Hence there are no integer solutions.

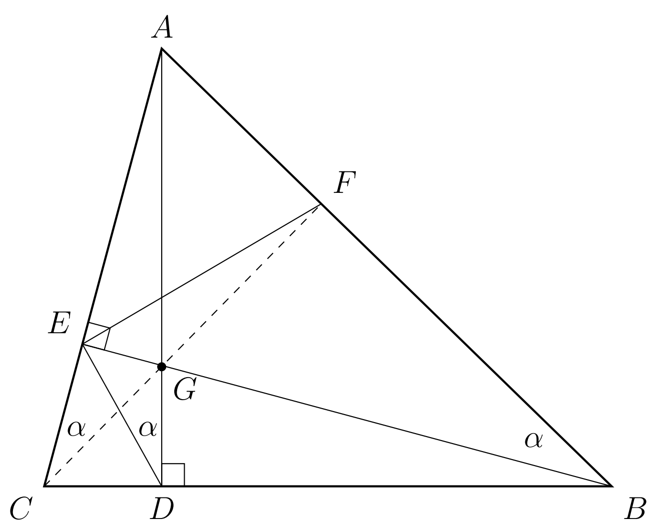
5. Let's label the vertices of our hexagon 1 through to 6. Then we can refer to edges as (ab) where a and b are vertices. Since we can re-label the hexagon however we want, let's just consider vertex 1. There are 5 edges coming out of vertex 1, and since we only have 2 colours, 3 of these edges must be the same colour, let's say red. And again, since we can re-label the hexagon however we want, let's suppose these edges are (12), (13) and (14). Suppose we don't have any red triangles, then (23) must be blue to prevent $\triangle 123$ from being red, also (34) must be blue to prevent $\triangle 134$ from being red, and (24) must be blue to prevent $\triangle 124$ from being red. But now (23), (34) and (24) are all blue, so $\triangle 234$ is blue.

Since it doesn't matter how you label the hexagon, this always happens.

In the case of the pentagon, colour the all the edges that are sides one colour (say blue) and all the edges that are diagonals the other colour (red). Since every triangle that can be made contains at least one side and one diagonal, there will be no purely blue or red triangles.



6. Let F be the point of intersection between CG extended and AB. Join EF and DE, as shown in the diagram.



Since AD and BE are altitudes $\angle BDA = \angle BEA = 90°$. Thus $BDEA$ is a cyclic quadrilateral. Consequently, $\angle EDA = \angle EBA = \alpha$.

Since $\angle CDG + \angle CEG = 180°$, $CDGE$ is also a cyclic quadrilateral. Thus $\angle ECG = \angle EDG = \alpha$. So $\angle FBE = \angle FCE$, and so $BCEF$ is a cyclic quadrilateral. Hence $\angle BFC = \angle BEC = 90°$, and so $CF \perp AB$, as required.

Senior Questions

1. (a) Taking square roots of both sides, we have

$$x = e^{-x/2}$$

(Note that we have tacitly assumed that $x > 0$ here.) Thus

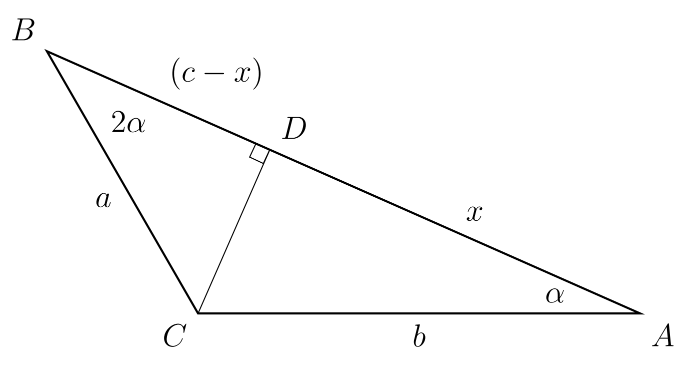
$$\begin{aligned} xe^{x/2} &= 1 \\ \frac{x}{2}e^{x/2} &= \frac{1}{2} \\ \therefore \frac{x}{2} &= W\left(\frac{1}{2}\right) \\ x &= 2W\left(\frac{1}{2}\right) \approx 0.7035 \quad \text{(to 4 sig fig)} \end{aligned}$$

So the point of intersection of the two curves is approximately $(0.7034, 0.4949)$.

(b)

$$\begin{aligned} x^x &= e \\ \ln x^x &= \ln(e) \\ x \ln x &= 1 \\ e^{\ln x} \cdot \ln x &= 1 \\ \therefore \ln x &= W(1) \\ x &= e^{W(1)} \approx 1.7632 \end{aligned}$$

2. Let $BC = a$, $AC = b$ and $AD = x$. Then $BD = c - x$.



By the sine rule,

$$\begin{aligned} \frac{\sin 2\alpha}{b} &= \frac{\sin \alpha}{a} \\ a \sin 2\alpha &= b \sin \alpha \\ 2a \sin \alpha \cos \alpha &= b \sin \alpha \\ b &= 2a \cos \alpha \end{aligned}$$

But $b = \dfrac{x}{\cos\alpha}$ and $a = \dfrac{c-x}{\cos 2\alpha}$, so

$$\frac{x}{\cos\alpha} = \frac{2(c-x)\cos\alpha}{\cos 2\alpha}$$

Collecting x's and α's to opposite sides of the equation, we obtain

$$\begin{aligned}\frac{x}{2(c-x)} &= \frac{\cos^2\alpha}{\cos 2\alpha} \\ &= \frac{\cos^2\alpha}{2\cos^2\alpha - 1}\end{aligned}$$

As $\alpha \to 0$, $\dfrac{\cos^2\alpha}{2\cos^2\alpha - 1} \to 1$, and so $x \to \dfrac{2c}{3}$.